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R-Symmetry and the Topological Twist of N=2 Effective Supergravities of Heterotic Strings[†]

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Abstract

We discuss R-symmetries in locally supersymmetric N=2 gauge theories coupled to hypermultiplets which can be thought of as effective theories of heterotic superstring models. In this type of supergravities a suitable R-symmetry exists and can be used to topologically twist the theory: the vector multiplet containing the dilaton-axion field has different R-charge assignments with respect to the other vector multiplets. Correspondingly a system of coupled instanton equations emerges, mixing gravitational and Yang–Mills instantons with triholomorphic hyperinstantons and axion-instantons. For the tree-level classical special manifolds $ST(n) = SU(1,1)/U(1) \times SO(2,n)/(SO(2) \times SO(n))$ R-symmetry with the specified properties is a continuous symmetry, but for the quantum corrected manifolds $\widehat{ST}(n)$ a discrete R–group of electric–magnetic duality rotations is sufficient and we argue that it exists.

1 Introduction

A large class of four dimensional topological field theories can be obtained from the topological twist of N=2 supergravity and N=2 matter theories [1–5]. The requirement that the twist should be well defined implies certain additional properties on the scalar manifold geometries, besides those imposed by N=2 supersymmetry, in order to obtain suitable ghost-number charges and in order that the quaternionic vielbein be a Lorentz vector after the twist. The needed properties pertain in particular to those scalar manifolds which emerge, at the tree level, in the effective theories of compactified superstrings. Specifically they are:

i) for the vector multiplet special manifold, an R-symmetry, which is essential to redefine the ghost number of the fields after the twist, and which, in the quantum case, is in general a discrete symmetry;

ii) for the hypermultiplet quaternionic manifold, an analogous "Q-symmetry", which permits a consistent redefinition of the Lorentz spin in the classical and quantum cases.

Continuous R–symmetries are common features of the coset manifolds which encode the local geometry of tree–level supergravity Lagrangians, emerging as effective theories of N=2 heterotic superstrings. The fact that the continuous symmetries present in the classical case break to discrete symmetries is suggested, on physical grounds, by the need to implement instanton corrections in the effective lagrangian. We will often refer to the tree level (classical) theory as to the "microscopic" theory, as it is done in paper [6] by Seiberg and Witten, in contrast to the quantum or effective "dynamical" theory, where loop and instanton corrections are taken into account and only the massless modes are included.

In this paper we are mainly concerned with the classical case, although we give indications of how our results can be generalized to the quantum case. Our main result, namely the structure of the instanton conditions that fix the topological symmetry, is independent of the detailed form of the theory and simply follows from the existence of a discrete or continuous R-symmetry with the properties we shall require. Hence the form of these instanton condition is universal and applies both to the classical and quantum case. Specifically it turns out that there are four equations describing the coupling of four types of instantons:

$$
\begin{array}{ll}
\text{i)} & \text{gravitational instanton} \\
\text{ii)} & \text{gauge–instantons} \\
\text{iii)} & \text{triholomorphic hyperinstantons} \\
\text{iv)} & \text{H-monopoles.}
\end{array}
\tag{1.1}
$$

Instanton equations of this type have already been discussed in [3, 4, 5, 7]; the main difference is that in [4, 5] the instanton conditions were only the first three of eq.s (1.1). The H-monopoles [8–12], namely the instanton-like configurations

$$
\partial_a D \;=\; \epsilon_{abcd} \mathrm{e}^D H^{bcd}
\tag{1.2}
$$

in the dilaton-axion sector were missing. In eq. (1.2) D is the dilaton field and $H_{\mu\nu\rho}$ is the curl of the antisymmetric axion tensor $B_{\mu\nu} : \partial_{[\rho}B_{\mu\nu]} = H_{\mu\nu\rho}$. The reason why they were missing in [4, 5] is the type of symmetry used there to define the ghost number, namely an on-shell R-duality based on the properties of the so-called minimal coupling. The new type of gravitationally extended R-symmetry that we present here is typically stringy in its origin and for the classical moduli–spaces is an ordinary off-shell symmetry, which does not mix electric and magnetic states as the R-duality of the minimal case does. In the quantum–corrected effective lagrangians R–symmetry reduces once again to an R–duality, namely to a discrete group of electric–magnetic duality rotations; yet the preferred direction of the dilaton–axion field is maintained also in the quantum case as it is necessary on physical grounds. The new version of R–symmetry discussed here provides the solution to several conceptual problems at the same time.

2   Outline and Philosophy

Four dimensional Topological Field Theories, which automatically select the appropriate instanton conditions, are derived by topologically twisting N=2, d=4 theories [1–5]. These latter include N=2 Yang–Mills theory, N=2 hypermultiplet sigma models, N=2 supergravity, or else the coupling of all such models together.

   In this paper we are concerned with the last case and with the special features of the topological theory that emerge when the parent N=2 matter coupled supergravity has the structure of a low energy Lagrangian for an N=2 heterotic string theory.

2.1   The geometry of vector multiplet and hypermultiplet scalar manifolds

As it is well known [13–27] the Lagrangian and the transformation rules of N=2 supergravity are completely determined in terms of the following geometrical data:

   1) The choice of a special Kähler manifold $\mathcal{SM}$ for the vector multiplet scalars

$$\dim_{\mathbb{C}}\mathcal{SM} = n + 1 \overset{\text{def}}{=} \# \text{ vector multiplets.} \tag{2.1}$$

   2) The choice of a quaternionic manifold QM for the hypermultiplet scalars

$$\dim_{\mathbb{Q}}\mathcal{QM} = \frac{1}{4}\dim_{\mathbb{R}}\mathcal{QM} = m \overset{\text{def}}{=} \# \text{ hypermultiplets.} \tag{2.2}$$

   3) The choice of a gauge group $\mathcal{G}$ with:

$$\dim_{\mathbb{R}}\mathcal{G} \leq n + 1 \tag{2.3}$$

that generates special isometries of $\mathcal{SM}$ and should have a triholomorphic action on the manifold $\mathcal{QM}$.

In this paper, we are concerned with the following choices:

$$\mathcal{SM} = ST(n) \stackrel{\text{def}}{=} \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2) \otimes SO(n)}$$

$$\mathcal{QM} = HQ(m) \stackrel{\text{def}}{=} \frac{SO(4,m)}{SO(4) \otimes SO(m)}$$

$$\mathcal{G} \subset SO(n) \tag{2.4}$$

where $\mathcal{G}$ is a n–dimensional subgroup of the $SO(n)$ appearing in the first equation above, such that:

$$\text{adjoint } \mathcal{G} = \text{vector } SO(n). \tag{2.5}$$

The structure given by eq. (2.4) is what one can obtain by certain N=2 truncations of N=4 matter coupled supergravity which, as it is well known, displays a unique coset structure:

$$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6, n+m)}{SO(6) \otimes SO(n+m)} \supset ST(n) \otimes HQ(m). \tag{2.6}$$

Other types of truncations can give different quaternionic coset manifolds $\mathcal{QM}$ [17], for instance $SU(2,m)/(SU(2) \times SU(m))$. Theories of type (2.4) originate, in particular cases, as tree-level low energy effective theories of the heterotic superstring compactified either on a $\mathbb{Z}_2$ orbifold of a six–torus $T^6/\mathbb{Z}_2$ or on smooth manifolds of $SU(2)$ holonomy, like $T_2 \otimes K3$ [19, 28, 29], else when the superstring is compactified on abstract free fermion conformal field theories [30–33] of type $(2,2)_{c=2} \oplus (4,4)_{c=6}$ [34]. Although in the following we focus on the particular case where $\mathcal{QM} = HQ(m)$, our discussion on R-symmetry is in fact concerned with the vector multiplet $ST(n)$ and applies also when $HQ(m)$ is replaced by other manifolds. Quantum corrections can change the geometry of $ST(n)$ or $HQ(m)$ in such a way that in the loop corrected Lagrangian they are replaced by new manifolds $\widehat{ST}(n)$ or $\widehat{HQ}(m)$, which are still respectively special Kählerian and quaternionic, but which can, in principle, deviate from the round shape of coset manifolds. It is known that in rigid Yang–Mills theories coupled to matter the hypermultiplet metric (which is hyperkählerian) does not receive quantum corrections neither perturbatively, nor non–perturbatively [35, 36, 37]. The same is true in N=2 supergravity theories derived from heterotic string theories: N=2 supersymmetry forbids a dilaton hypermultiplet mixing [36, 38, 39] since the dilaton is the scalar component of a vector multiplet. Hence in this case, while the scalar manifold is replaced by $\widehat{ST}(n)$, the quaternionic manifold $HQ(m)$ is unmodified.

The reverse is true (i.e. there are no quantum corrections to the vector multiplet metric) for N=2 supergravities derived from type II strings [40]

Generically continuous isometries break to discrete ones. This may be a consequence both of $\mathcal{O}(\alpha')$ corrections due to the finite size of the string (discrete t–dualities generated by non–perturbative world–sheet effects) and of non–perturbative

quantum effects due to space–time instantons (discrete Peccei Quinn axion symmetries). Furthermore it can either happen that the discrete quantum symmetries are just restrictions to special values of the parameters of the classical continuous symmetries or that they are entirely new ones. Usually the first situation occurs when the local quantum geometry coincides with the local classical geometry, namely when there are no corrections to the moduli space–metric except for global identifications of points, while the second situation occurs when not only the global moduli geometry, but also the local one is quantum corrected. As we have stressed, although $\widehat{ST}(n)$ and $\widehat{HQ}(m)$ may be quite different manifolds from their tree level counterparts, they should still possess an R-symmetry or a Q–symmetry so that the topological twist may be defined. Let us then discuss, on general grounds, the problems related to the twist of matter coupled N=2 supergravity.

2.2 The topological twist and the problem of ghost numbers

In his first paper on topological field theories [1], Witten had shown how to derive a topological reinterpretation of N=2 Yang–Mills theory in four–dimensions by redefining the Euclidean Lorentz group:

$$SO(4)_{spin} = SU(2)_L \otimes SU(2)_R \qquad (2.7)$$

in the following way:

$$SO(4)'_{spin} = SU(2)_L \otimes SU(2)'_R \; ; \qquad SU(2)'_R = \mathrm{diag}\,(SU(2)_I \otimes SU(2)_R) \quad (2.8)$$

where $SU(2)_I$ is the automorphism group of N=2 supersymmetry. In order to extend Witten's ideas to the case of an arbitrary N=2 theory including gravity and hypermultiplets, four steps, that were clarified in refs. [3, 4], are needed:

i) Systematic use of the BRST quantization, prior to the twist.

ii) Identification of a gravitationally extended R-symmetry that can be utilized to redefine the ghost–number in the topological twist.

iii) Further modification of rule (2.8) for the redefinition of the Lorentz group that becomes:

$$SO(4)'_{spin} = SU(2)'_L \otimes SU(2)'_R \qquad \begin{cases} SU(2)'_R = \mathrm{diag}\,(SU(2)_I \otimes SU(2)_R) \\ SU(2)'_L = \mathrm{diag}\,(SU(2)_Q \otimes SU(2)_L) \end{cases}$$
$$(2.9)$$

Here $SU(2)_Q$ is a group whose action vanishes on all fields except on those of the hypermultiplet sector, so that its role was not perceived in Witten's original case.

iv) Redefinition of the supersymmetry ghost field (*topological shift).*

Points i) and iv) of the above list do not impose any restriction on the scalar manifold geometry, so we do not discuss them further, although we shall use the concept of topological shift in later sections. (We refer the reader to [3, 4] for further details). Points ii) and iii), on the other hand have a bearing on the geometry of $\widehat{ST}(n)$ and $\widehat{HQ}(m)$ and are our main concerns.

Topological field theories are cohomological theories of a suitable BRST complex and as such they need a suitable ghost number q_{gh} that, together with the form degree, defines the double grading of the double elliptic complex. In the topological twist, at the same time with the spin redefinition (2.9) one has a redefinition of the BRST charge and of the ghost number, as follows:

$$
\begin{aligned}
Q'_{BRST} &= Q_{BRST} + \overset{0}{Q^{-}_{BRST}} \\
q'_{gh} &= q_{gh} + q_R.
\end{aligned} \tag{2.10}
$$

Here Q_{BRST} is the old BRST charge that generates the BRST transformations of the N=2 matter coupled supergravity and q_{gh} is the old ghost number associated with the BRST complex generated by Q_{BRST}. We discuss now the shifts $\overset{0}{Q^{-}_{BRST}}$ and q_R, beginning with the former. The whole interest of the topological twist is that $\overset{0}{Q^{-}_{BRST}}$ is just a component of the Wick–rotated supersymmetry generators. It is defined as follows.

Writing the N=2 Majorana supercharges in the following bi-spinor notation:

$$
\mathbb{Q}_A = \begin{pmatrix} Q^{\alpha A} \\ Q_{\dot\alpha A} \end{pmatrix} \qquad \begin{cases} \alpha = 1, 2 \\ \dot\alpha = \dot1, \dot2, \end{cases} \tag{2.11}
$$

so that a transformation of spinor parameter χ_A is generated by:

$$
\chi \cdot \mathbb{Q} = \chi_{\alpha A} \, Q^{\alpha A} + \chi^{\dot\alpha A} \, Q_{\dot\alpha A}, \tag{2.12}
$$

we can perform the decomposition:

$$
Q_{\dot\alpha A} = \epsilon_{\dot\alpha A} \, \overset{0}{Q^{-}_{SUSY}} + \left(\sigma_x \, \epsilon^{-1}\right)_{\dot\alpha A} \, \overset{x}{Q^{-}_{SUSY}} \tag{2.13}
$$

and identify $\overset{0}{Q^{-}_{SUSY}}$ with the shift of the BRST charge introduced in eq. (2.10). It has spin zero as a BRST charge should have. In eq (2.13) σ_x are the standard Pauli matrices and $\epsilon_{AB} = -\epsilon_{BA}$, with $\epsilon_{12} = 1$. Eq. (2.13) makes sense because of the twist. Indeed, after $SU(2)_R$ has been redefined as in eq.(2.9) the isotopic doublet index A labeling the supersymmetry charges becomes an ordinary dotted spinor index.

Let us now come to the discussion of the ghost number shift.

2.3 R-symmetry in rigid N=2 theories

The topological twist of a rigid N=2 supersymmetric Yang–Mills theory yields topological Yang–Mills theory, where the fields of the N=2 supermultiplet have the

following reinterpretation:

$$
\begin{aligned}
\text{gauge boson} \ \ A^\alpha_\mu \ &\rightarrow \ \text{phys. field} \ \ q_{gh} = 0 \\
\text{left--handed gaugino} \ \ \lambda^{\alpha A} \ &\rightarrow \ \text{top. ghost} \ \ q_{gh} = 1 \\
\text{right--handed gaugino} \ \ \lambda^A_{\alpha\star} \ &\rightarrow \ \text{top. antighost} \ \ q_{gh} = -1 \\
\text{scalar} \ \ Y^I \ &\rightarrow \ \text{ghost for ghosts} \ \ q_{gh} = 2 \\
\text{conjug. scalar} \ \ \overline{Y}^{I^\star} \ &\rightarrow \ \text{antighost for antighosts} \ \ q_{gh} = -2 \quad (2.14)
\end{aligned}
$$

Hence, for consistency, the N=2 Yang–Mills theory should have, prior to the twist, a global $U(1)$ symmetry with respect to which the fields have charges identical with the ghost numbers they acquire after the twist. In the minimal coupling case such a symmetry does indeed exist and it is named R-symmetry (see for example [41]).

By minimal coupling we mean the situation where the *rigid special geometry* [6, 42] of the scalar manifold is defined by the following generating function of quadratic type:

$$
F(Y) = \mathrm{i} g^{(K)}_{IJ} Y^I Y^J \tag{2.15}
$$

where the Y^I scalar fields are identified with the rigid special coordinates and g^K_{IJ} is the constant Killing metric of the gauge group $\mathcal{G}$. With such a choice, the Kähler metric of the Y-scalar σ-model, $\mathcal{L}^{scalars}_{kin} = g_{IJ^\star}(Y, \overline{Y})\partial_\mu Y^I \partial_\mu \overline{Y}^{J^\star}$ defined by:

$$
\begin{aligned}
g_{IJ^\star}(Y, \overline{Y}) &= \partial_I \, \partial_{J^\star} \, \mathcal{K}_{rigid}(Y, \overline{Y}) \\
&= \mathrm{i} \partial_I \, \partial_{J^\star} \left(Y^L \, \partial_{L^\star}\overline{F} - \overline{Y}^{L^\star} \, \partial_L F \right) \\
&= 2\mathrm{Im}\partial_I \, \partial_J \, F(Y) \tag{2.16}
\end{aligned}
$$

takes the constant value:

$$
g_{IJ^\star} = g^{(K)}_{IJ} \tag{2.17}
$$

and the kinetic term $\mathcal{L}^{vector}_{kin} = \frac{1}{2\mathrm{i}}[\mathcal{N}_{IJ}(\overline{Y}) \, F^{+I}_{\mu\nu} \, F^{+J}_{\mu\nu} - \overline{\mathcal{N}}_{IJ}(Y) \, F^{-I}_{\mu\nu} \, F^{-J}_{\mu\nu}]$ for the vectors, whose general definition is provided by:

$$
\overline{\mathcal{N}}_{IJ}(Y) = \partial_I \partial_J F(Y) \tag{2.18}
$$

is also of the standard form required for a renormalizable gauge theory:

$$
\mathcal{L}^{vector}_{kin} = -g^{(K)}_{IJ} \left(F^{+I}_{\mu\nu} F^{+J}_{\mu\nu} + F^{-I}_{\mu\nu} F^{-J}_{\mu\nu} \right). \tag{2.19}
$$

Minimal coupling corresponds, in the language of reference [6], to the *microscopic gauge theory*. This theory has a scalar potential of the form:

$$
V(Y, \overline{Y}) \propto g^{(K)}_{IJ} \, f^I_{\ RS} \, f^J_{\ LM} \, Y^R \overline{Y}^S \, Y^L \overline{Y}^M \tag{2.20}
$$

where f^I_{JK} are the structure constants of the group $\mathcal{G}$. The scalar potential has *flat directions*, namely it vanishes for arbitrary values of

$$
\{Y\} \in \mathbb{C} \otimes \mathcal{H} \tag{2.21}
$$

$\mathcal{H}$ being the Cartan subalgebra of $\mathcal{G}$. If we denote by Y^α the scalar fields in the $\mathcal{H}$–subalgebra, then Y^α are the moduli of the spontaneously broken gauge theory that has $\mathcal{H}$ as unbroken gauge group and the components A_μ^α of the $\mathcal{H}$–connection as massless gauge fields. The effective low energy lagrangian for the massless modes is no longer a minimally coupled N=2 gauge theory. Indeed its structure is determined by a *rigid special geometry* encoded in a generating function of the following form:

$$\widehat{F}(Y^\alpha) \ = \ \mathrm{i}g_{\alpha\beta}Y^\alpha Y^\beta \ + \ \delta\widehat{F}(Y^\alpha) \tag{2.22}$$

where $g_{\alpha\beta}$ is the Cartan matrix of $\mathcal{G}$ and $\delta\widehat{F}(Y^\alpha)$ accounts for the unique one–loop correction and for the infinite sum of the instanton corrections [43]. In the topologically twisted theory the deviation of the prepotential $F(Y)$ from the quadratic form corresponds to perturbing the original minimal topological lagrangian by means of all the available topological observables, namely

$$S_{\min} \ \longrightarrow \ S_{\min} \ + \ \sum_k c(P_k) \int_{\mathcal{M}_4} \Phi_{4,4k-4}(P_k)$$

$$\Phi_{4,4k-4}(P_k) \ = \ 4\text{–form part of } P_k(\widehat{F}) \tag{2.23}$$

where $P_k(F)$ is any invariant polynomial of order k of the gauge Lie algebra (i.e. a characteristic class) and where by $\widehat{F}$ we have denoted the ghost–extended field–strength according to the standard rules of topological gauge theories in the Baulieu–Singer set up [44, 3, 4, 45]. From this point of view the vacuum expectation values of the ghost fields Y^α play the role of *ghost–charged* topological coupling constants. The continuous R-symmetry group $U(1)_R$ of the minimally coupled theory is now broken, but either a discrete subgroup $\widehat{G}_R \subset U(1)_R$ survives or a new discrete R–group $\widehat{G}_R^{quantum}$ replaces it.

A similar phenomenon should occur in the gravitational case and this is the matter of our study in the sequel.

2.4 R-symmetry in local N=2 theories and the moduli spaces of gravitational instantons

This being the situation in the rigid case, it is clear that, when N=2 supersymmetry is made local, R-symmetry should extend to a suitable symmetry of matter coupled supergravity. This problem was addressed in [4], where it was shown that the minimally coupled local theory, which is also based on a quadratic generating function of the *local Special Geometry*:

$$F\left(X^0, X^\alpha\right) \ = \ i\left[\left(X^0\right)^2 - \sum_{\alpha=1}^{n}\left(X^\alpha\right)^2\right], \tag{2.24}$$

and which corresponds to the following choice for $\mathcal{SM}$:

$$\mathcal{SM} \ = \ \frac{SU(1,n)}{U(1) \otimes SU(n)} \tag{2.25}$$

possesses an *R-duality*, namely an extension of *R*-symmetry that acts as a duality rotation on the graviphoton field strength,

$$\begin{aligned}
\delta F_{grav}^{+ab} &= e^{i\theta} F_{grav}^{+ab} \\
\delta F_{grav}^{-ab} &= e^{-i\theta} F_{grav}^{-ab}
\end{aligned} \tag{2.26}$$

mixing therefore electric and magnetic states. This result enabled the authors of [4] to discuss the topological twist in the case where the choice (2.25) is made.

In this paper we show that the string inspired choice of eq. (2.4) yields another form of *R*-symmetry that allows the topological twist to be performed also in this case. Actually the new form of *R*-symmetry displays a new important feature that leads to the solution of a problem left open in the previous case.

In the case (2.25) all the vector fields, except the graviphoton, are physical since they have zero *R*-charge and hence zero ghost number after the twist. On the contrary, in this case, all the vector multiplet scalar fields are ghost charged and hence unphysical. The limit of pure topological gravity is obtained by setting $n = 0$ in eq. (2.25). This definition of 4D topological gravity [3] is correct but has one disadvantage that we briefly summarize. The topological observables of the theory

$$\int_{\mathcal{C}_2} \Phi_{(2,4n-2)} = \int_{\mathcal{C}_2} \text{Tr} \left(\widehat{R} \wedge \widehat{R} \wedge \ldots \wedge \widehat{R} \right)_{(2,4n-2)} \tag{2.27}$$

(where $\mathcal{C}_2$ is a two cycle) have a ghost number which is always even being obtained from the trace of the product of an *even* number of (extended) curvature 2-forms (that this number should be even is a consequence of the self–duality of R^{ab} in instanton backgrounds). On the other hand the moduli space of a typical gravitational instanton (an ALE manifold) has a moduli space with dimensionality [12, 46, 47, 48, 49]:

$$\dim_{\mathbb{C}} \mathcal{M}_{moduli}(ALE) = 3\tau \tag{2.28}$$

τ being the Hirzebruch signature. It appears therefore difficult to saturate the sum rule

$$\sum_{i=1}^{n} gh^i = 3\tau \tag{2.29}$$

needed for the non-vanishing of an *n*-point topological correlator of local observables. Notice, however, that it is possible to find nontrivial topological correlation functions, satisfying the selection rule (2.29), between non local observables of the form $\int_{\mathcal{C}_1} \Phi_{(1,4n-1)}$ for the topological gravity with the Eguchi–Hanson instanton [50].

The origin of this problem is fairly evident to the string theorist and in particular to the string theorist who has experience with Calabi–Yau compactifications. Let us see why. The number 3τ emerges as the number of deformations of the self-dual metrics on the ALE–manifold. To each self-dual harmonic 2-form one attaches a complex parameter (and hence 2 real parameters) for the deformations of the complex structure and a real parameter for the deformations of the Kähler structure,

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which sum to three parameters times the Hirzebruch signature. This counting, appropriate to pure gravity, is incomplete in the effective theory of superstrings where one has also the axion and the dilaton, besides the metric. An additional real modulus is associated with each selfdual 2-form for the deformations of the axion. This parameter can be used to complexify the complex structure deformations making the total dimension of moduli space 4τ rather than 3τ. Hence a sound 4-dimensional topological gravity should include also the dilaton and the axion, as suggested by the superstring. In the N=2 case these two fields are combined together into the complex field S, which is just the scalar field of an additional vector multiplet. Therefore we would like a situation where of the $n+1$ vector multiplets coupled to supergravity, n have the ghost numbers displayed in eq. (2.14), while one behaves in the reversed manner, namely:

$$
\begin{aligned}
\text{gauge boson } A_\mu^s &\rightarrow \text{ghost. for ghost} & q_{gh} &= 2 \\
\text{left–handed gaugino } \lambda^{sA} &\rightarrow \text{top. antighost} & q_{gh} &= -1 \\
\text{right–handed gaugino } \lambda_{s\star}^A &\rightarrow \text{top. ghost} & q_{gh} &= 1 \\
\text{scalar } S &\rightarrow \text{phys. field} & q_{gh} &= 0 \\
\text{conjug. scalar } \overline{S} &\rightarrow \text{phys. field} & q_{gh} &= 0.
\end{aligned}
\tag{2.30}
$$

This phenomenon is precisely what takes place in the new form of R-duality, which is actually an R-symmetry, which applies to the classical manifold $ST(n)$.

The proof of this statement is one of the main points of the present paper.

In the quantum case we should require that the same R-charge assignments (2.14) and (2.30) holds true. For this to be true it suffices, as stressed in the introduction, that only a (suitable) discrete R-symmetry survives.

2.5 Gravi–Matter Coupled Instantons

Provided the above restrictions on the scalar manifolds are implemented one can describe in general terms the coupled matter, gauge and gravitational instantons that arise from the topological twist by means of the following equations:

$$
\begin{aligned}
R^{-ab} - \sum_{u=1}^{3} J_u^{-ab} q^\star \widehat{\Omega}^{-u} &= 0 \\
\partial_a D - \epsilon_{abcd} \mathrm{e}^D H^{bcd} &= 0 \\
\mathcal{F}^{-\alpha\, ab} - \frac{g}{2 \exp D} \sum_{u=1}^{3} J_u^{-ab} \mathcal{P}_\alpha^{-u} &= 0 \\
\mathcal{D}_\mu q^P - \sum_{u=1}^{3} (j_u)_\mu^{\ \nu} \mathcal{D}_\nu q^Q (J_u)_Q^{\ P} &= 0.
\end{aligned}
\tag{2.31}
$$

In the above equations R^{-ab} is the antiselfdual part of the Riemann curvature 2–form (a, b are indices in the tangent of the space time manifold), $q^\star \widehat{\Omega}^{-u}$ denotes the

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pull–back, via a gauged–triholomorphic map:

$$q \, : \, M_{space-time} \, \longrightarrow \, HQ(m) \qquad (2.32)$$

of the "gauged" 2–forms $\widehat{\Omega}^{-\,u}$ corresponding to one of the two quaternionic structures of $HQ(m)$ (see appendices A and C)[1]. $\mathcal{P}_\alpha^{-\,u}$ are the corresponding momentum map functions for the triholomorphic action of the gauge group $\mathcal{G}$ on $HQ(m)$. Furthermore J_u^{-ab} is nothing else but a basis of anti-selfdual matrices in $\mathbb{R}^4$. The second of equations (2.31) describes the H–monopole or *axion–dilaton instanton* first considered by Rey in [8] and subsequently identified with the Regge–D'Auria torsion instantons [9] and also with the semi–wormholes of Callan et al [10] according to the analysis of [12]. In the Rey formulation, that is the one appearing here, the H– monopoles have vanishing stress–energy tensor, so that they do not interfere with the gravitational instanton conditions. The last of eq.s (2.31) is the condition of triholomorphicity of the map (2.32) rewritten with covariant rather than with ordinary derivatives. Such triholomorphic maps are the four–dimensional σ–model instantons, or *hyperinstantons* [4, 5]. Finally, in the same way as the first of eq.s (2.31) is the deformation of the gravitational instanton equation due to the presence of hyperinstantons, the third expresses the modification of Yang–Mills instantons due to the same cause. The space–time metric is no longer self–dual yet the antiself–dual part of the curvature is just expressed in terms of the hyperinstanton quaternionic forms. The same happens to the antiself–dual part of the Yang–Mills field strength. Deleting the first three of eq.s (2.31) due the gravitational interactions one obtains the appropriate generalization to any gauge–group and to any matter sector of the so called monopole–equations considered by Witten in [7]. That such equations were essentially contained in the yield of the topological twist, as analysed in [4], was already pointed out in [5]. The main novelty here is the role played by the dilaton–axion sector that, as already emphasized, should allow the calculation of non–vanishing topological correlators between local observables as intersection numbers in a moduli–space that has now an overall complex structure.

2.6 Topological gauge fixing as supersymmetric backgrounds

To find supersymmetric backgrounds of a supersymmetric theory, one usually looks for solutions of the eq.s:

$$\delta_{SUSY}\psi \;=\; 0 \qquad (2.33)$$

where ψ is any fermion of the theory. In a generic $N=2$ theory which includes the dilaton there are four types of fermion, namely the gravitino, the dilatino, the gauginos and the hyperinos. Correspondingly there are four sets of differential equations to be satisfied by the bosonic backgrounds. In euclidean signature there

---

[1]Notice that the instanton equation have the same expression also in the quantum case $\widehat{ST(n)}, \widehat{HQ(m)}$

exist non trivial solutions that preserve at most half of the supersymmetries, namely those with supersymmetry parameter of a given chirality. A set of solutions of these equations is obtained precisely by solving eq.s (2.31). The reason for this is that the instanton conditions (2.31) are the BRST–variations of the topological antighosts which in the untwisted version of the theory coincide with the supersymmetry variations of the fermions of a given chirality. The only difference is that in eq.s (2.31), in addition to the fermions we have set to zero also those bosonic fields that have non–zero ghost number, namely $Y^\alpha, A^s_\mu, A^0_\mu$.

3 Special geometry of the $ST(n)$ manifolds

In this section we recall some general properties of special Kähler manifolds [13, 14, 15, 22, 24, 25, 27], then we focus on the manifolds $ST(n)$ listed in eq. (2.4).

Special geometry is the natural geometric structure which arises in the coupling of N=2 four dimensional supergravity to vector multiplets. In particular, when N=2 supergravity is regarded as an effective theory for the massless modes of the compactified heterotic string, the vector multiplets have a well defined structure. Fixing their number to be $n+1$ we have that n of them contain the ordinary gauge vectors:

$$(A^\alpha, \lambda^{\alpha A}, \lambda^{\overline{\alpha}}_A, Y^\alpha), \quad \alpha = 1, \ldots, n \tag{3.1}$$

and one:

$$(A^S, \lambda^{SA}, \lambda^{\overline{S}}_A, S) \tag{3.2}$$

contains the dilaton-axion field:

$$
\begin{aligned}
S &= \mathcal{A} + ie^D \\
\nabla_\sigma \mathcal{A} &= \frac{\epsilon_{\sigma\mu\nu\rho}}{\sqrt{|g|}} e^{2D} H^{\mu\nu\rho} = \frac{\epsilon_{\sigma\mu\nu\rho}}{\sqrt{|g|}} e^{2D} \partial^\mu B^{\nu\rho}.
\end{aligned}
\tag{3.3}
$$

In eq.s (3.1) and (3.2) A denotes the gauge connection 1-form, λ^A and λ_A denote the left-handed and the right-handed parts of the gauginos $(\gamma_5 \lambda^A = \lambda^A, \gamma_5 \lambda_A = -\lambda_A)$ and (S, Y^α) are the complex scalar fields of the corresponding multiplets. The N=2 supersymmetry imposes specific constraints on the manifold spanned by the scalar fields. This manifold must be a Hodge–Kähler manifold of restricted type, namely a special Kähler manifold [13, 14, 22, 25].

In a generic $n+1$-dimensional special Kähler manifold, the Kähler two form can be expressed by the formula

$$K = \frac{i}{2\pi} \partial\overline{\partial} \log ||W(S,Y)||^2 = -\frac{i}{2\pi} \partial\overline{\partial} \log ||\Omega||^2 \tag{3.4}$$

where $W(S,Y)$ is a holomorphic section of the Hodge line bundle $\mathcal{L}_H \xrightarrow{\pi} \mathcal{SM}$ and

$$\Omega = (X^\Lambda, F_\Lambda) \tag{3.5}$$

is a holomorphic section of $\mathcal{L}_H^2 \times \mathcal{SP}$, where $\mathcal{SP} \xrightarrow{\pi'} \mathcal{SM}$ is a flat, rank $2n+4$-vector bundle, with $SP(2n+4, \mathbb{R})$ structural group. This amounts to say that the Kähler potential $\mathcal{K}$ has the following expression:

$$\mathcal{K} = -\log\left[-\mathrm{i}(\overline{X}, \overline{F})^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix} \right]. \tag{3.6}$$

The symplectic index Λ runs over $n+2$ values, and in the cases related to string compactifications it has the following labels: $\{0, S, \alpha\}$ ($\alpha = 1, \ldots n$), the index zero being associated to the gravitational multiplet.

In many bases, but not necessarily in all bases, the symplectic section (3.5) can be chosen in such a way that

$$F_\Lambda = \frac{\partial F(X)}{\partial X^\Lambda} \tag{3.7}$$

where $F(X)$ is a degree two homogeneous function of the X^Λ coordinates, named the prepotential.

Eq.s (3.4) and (3.6) implies that the Riemann tensor for a generic special manifold satisfies the following identity :

$$R_{ij^*lk^*} = g_{ij^*}g_{lk^*} + g_{ik^*}g_{lj^*} - C_{ilp}C_{j^*k^*p^*}g^{pp^*} \tag{3.8}$$

where $C_{ijk} = e^{\mathcal{K}}W_{ijk}$ are suitable sections of $\mathcal{L}_H^2 \times [T^{(1,0)}\mathcal{SM}]^3$. These sections have a double physical interpretation. In the N=2 effective lagrangians they play the role of fermionic anomalous magnetic moments , while, in the associated N=1 theories (obtained from the N=2 ones via the h–map [51, 34, 52]), they can be interpreted as Yukawa couplings.

The elements of the symplectic structural group $SP(2n+4, \mathbb{R})$, namely matrices with the following block structure

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \begin{cases} A^T C - C^T A = 0 \\ B^T D - D^T B = 0 \\ A^T D - C^T B = \mathbf{1} \end{cases} \tag{3.9}$$

induce coordinate transformations on the scalar manifold while acting, at the same time, as duality rotations on the symplectic vector of electric and magnetic field strengths [2]:

$$(\mathcal{F}_{ab}^{-\Lambda}, G_{\Lambda ab}^-) \qquad \text{where } G_{\Lambda ab}^- = -\mathrm{i}\frac{\delta\mathcal{L}}{\delta\mathcal{F}_{ab}^{-\Lambda}}, \tag{3.10}$$

In the case the scalar manifold $\mathcal{SM}$ admits a continuous or discrete isometry group G_{iso}, this group must be suitably embedded into the duality group $SP(2n+4, \mathbb{R})$

---

[2]The $\mathcal{F}_{ab}$ are the components along the space-time vierbeins of the field-strength $\mathcal{F}$ and $\mathcal{F}_{ab}^\pm$ their (anti)selfdual projections. The (anti)selfdual parts satisfy $\epsilon_{abcd}\mathcal{F}^{\pm cd} = \pm 2\mathrm{i}\mathcal{F}_{ab}^\pm$ and are defined by $\mathcal{F}^{\pm ab} = \frac{1}{2}\left(\mathcal{F}^{ab} \pm \widetilde{\mathcal{F}}^{ab}\right)$ where $\widetilde{\mathcal{F}}^{ab} = -\frac{\mathrm{i}}{2}\epsilon^{abcd}\mathcal{F}_{cd}$ is the dual tensor.

and the corresponding duality rotations, induced by the embedding, leave form invariant the system of Bianchi identities plus equation of motion [53].

In this paper we are mainly concerned with the case $\mathcal{SM} = ST(n)$, and, in the sequel, we focus our attention to its particular properties.

The special Kähler manifold $ST(n)$ has been studied using different parametrizations, corresponding to different embeddings of the isometry group $SL(2, \mathbb{R}) \times SO(2, n)$ into the symplectic group $SP(2n + 4, \mathbb{R})$. The first studied parametrization was based on a cubic type prepotential $F(X) = \frac{1}{X^0} X^S X^r X^t \eta_{rt}$, where η_{rt} is the constant diagonal metric with signature $(+, -, \ldots, -)$ in a n-dimensional space [15]. In this parametrization only an $SO(n - 1)$ subgroup of $SO(2, n)$ is linearly realized and it is possible to gauge only up to $n - 1$ vector multiplets. This means that, of the n ordinary gauge vectors sitting in the n vector multiplets, only $n - 1$ can be gauged.

From a string compactification point of view one does not expect this restriction: it should be possible to gauge all the n vector multiplets containing the ordinary gauge vectors A^α. This restriction motivated the search for a second parametrization, where the $SO(n)$ subgroup is linearly realized. This parametrization is based on the "square root" prepotential $F(X) = \sqrt{(X_0^2 + X_1^2) X^\alpha X^\alpha}$ [26].

However, in principle, it should be possible to find a linear realization of the full $SO(2, n)$ group, as it is predicted by the tree level string symmetries. In this case one can also gauge the graviphoton and the gauge field associated to the dilaton multiplet. This is explicitly realized in a recent work [36], where the new parametrization of the symplectic section is based on the following embedding of the isometry group $SO(2, n) \times SL(2, \mathbb{R})$ into $SP(2n + 4, \mathbb{R})$.

$$A \in SO(2, n) \qquad \hookrightarrow \quad \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \eta A \eta^{-1} \end{pmatrix} \in Sp(2n + 4, \mathbb{R})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \quad \hookrightarrow \quad \begin{pmatrix} a\mathbb{1} & b\eta^{-1} \\ c\eta & d\mathbb{1} \end{pmatrix} \in Sp(2n + 4, \mathbb{R}) \,,$$

(3.11)

where $A^T \eta A = \eta$. Notice that, in this embedding, the $SO(2, n)$ transformations, when acting on the section $(\mathcal{F}_{ab}^{-\Lambda}, G_{\Lambda ab}^-)$, do not mix the $\mathcal{F}$ with the G's. Thus the true duality transformations mixing the equations of motion and Bianchi identities are generated by the embedding of the $SU(1, 1)$ factor only, so that the field S, that in our case parametrizes the coset $SU(1, 1)/U(1)$, plays a very different role from the Y^α fields.

The explicit form of the symplectic section corresponding to the embedding of eq. (3.11) is:

$$(X^\Lambda, F_\Lambda) \;\; = \;\; (X^\Lambda, S\eta_{\Lambda\Sigma} X^\Sigma)$$

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$$X^\Lambda = \begin{pmatrix} 1/2\,(1+Y^2) \\ i/2\,(1-Y^2) \\ Y^\alpha \end{pmatrix}. \tag{3.12}$$

In eq. (3.12) Y^α are the Calabi–Visentini coordinates, parametrizing the coset manifold $SO(2,n)/SO(2) \times SO(n)$. The pseudoorthogonal metric $\eta_{\Lambda\Sigma}$ has the signature $(+,+,-,\ldots,-)$.

Notice that, with the choice (3.12), it is not possible to describe F_Λ as derivarivatives of any prepotential. The Kähler potential for $ST(n)$ is obtained inserting in eq. (3.6) the explicit form of the section (3.12), namely:

$$\mathcal{K} = \mathcal{K}_1(S,\overline{S}) + \mathcal{K}_2(Y,\overline{Y}) = -\log i(\overline{S} - S) - \log \overline{X}^T \eta X. \tag{3.13}$$

From eq. (3.13) it easy to see that the Kähler metric has the following block diagonal structure:

$$\begin{pmatrix} g_{S\overline{S}} & \mathbf{0} \\ \mathbf{0} & g_{\alpha\beta^*} \end{pmatrix} \qquad \begin{cases} g_{S\overline{S}} = \partial_S \partial_{\overline{S}} \mathcal{K}_1 = \frac{-1}{(\overline{S}-S)^2} \\ g_{\alpha\beta^*}(Y,\overline{Y}) = \partial_\alpha \partial_{\beta^*} \mathcal{K}_2. \end{cases} \tag{3.14}$$

The explicit expression of $g_{\alpha\overline{\beta}}(Y,\overline{Y})$ is not particularly relevant for our purposes. In the sequel, while discussing the instanton conditions, we will be interested only in its value at $Y = 0$ ($\overline{Y} = 0$):

$$g_{\alpha\beta^*}(Y=0) = 2\delta_{\alpha\beta^*}. \tag{3.15}$$

The connection one form Q of the line bundle $\mathcal{L}_H$ is expressed in terms of the Kähler potential as

$$Q^{(1,0)} + Q^{(0,1)} = \frac{1}{2i}[\partial_S \mathcal{K} dS + \partial_\alpha \mathcal{K} dY^\alpha] + c.c. \tag{3.16}$$

The explicit value of $Q^{(1,0)}$ at $Y = 0$ is

$$Q^{(1,0)}(Y=0) = \frac{1}{2}\frac{dS}{\overline{S} - S}. \tag{3.17}$$

The anomalous magnetic moments-Yukawa couplings sections C_{ijk} ($i = S, \alpha$) have a very simple expression in the chosen coordinates:

$$C_{S\alpha\beta} = -e^{\mathcal{K}}\delta_{\alpha\beta}, \tag{3.18}$$

all the other components being zero.

In a general N=2 supergravity coupled to vector multiplets the lagrangian for the vector bosons has a structure generalizing the rigid expression, namely

$$\begin{aligned} \mathcal{L}_{\text{kin}} &\propto \frac{1}{2i}(\mathcal{N}_{\Lambda\Sigma}\mathcal{F}_{ab}^{+\Lambda}\mathcal{F}_{ab}^{+\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}_{ab}^{-\Lambda}\mathcal{F}_{ab}^{-\Sigma}) \\ &= \frac{1}{2}(\text{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}_{ab}^\Lambda\mathcal{F}_{ab}^\Sigma - i\text{Re}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}_{ab}^\Lambda\widetilde{\mathcal{F}}_{ab}^{+\Sigma}). \end{aligned} \tag{3.19}$$

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The general form of the matrix $\mathcal{N}_{\Lambda\Sigma}$ in the cases in which the prepotential F exists is given in [14, 15, 25]. Its further generalization, including also the cases where F does not exists, has been found in [36]. In our specific case, $\mathcal{N}_{\Lambda\Sigma}$ is given by:

$$\mathcal{N}_{\Lambda\Sigma} = (S - \overline{S})\frac{X_\Lambda \overline{X}_\Sigma + \overline{X}_\Lambda X_\Sigma}{\overline{X}^T \eta X} + \overline{S}\eta_{\Lambda\Sigma}. \tag{3.20}$$

In particular we have that $\operatorname{Re}\mathcal{N}_{\Lambda\Sigma} = \operatorname{Re} S\eta_{\Lambda\Sigma} = \mathcal{A}\eta_{\Lambda\Sigma}$. Moreover, at $Y = 0$, the only non-zero components of $\operatorname{Im}\mathcal{N}_{\alpha\beta}$ are given by

$$\operatorname{Im}\mathcal{N}_{\alpha\beta}(Y = 0) = \operatorname{Im} S\, \delta_{\alpha\beta} = \exp D\, \delta_{\alpha\beta}. \tag{3.21}$$

Thus at $Y = 0$ the kinetic term for the ordinary gauge vectors A^α reduces to $\frac{\operatorname{Im} S}{g^2}\mathcal{F}_{ab}^\alpha \mathcal{F}_{ab}^\alpha$, where we have explicitly taken into account the gauge coupling dependence, via the usual redefinition $A^\alpha \to \frac{1}{g}A^\alpha$. This means that we can reinterpret $g_{\mathrm{eff}} = \frac{g}{\sqrt{\operatorname{Im}S}}$ as the effective gauge coupling.

4 R-symmetry in N=2 Supergravity

In this section we give the general definition of gravitationally extended R-symmetry. Such a definition in the continuous case pertains to the $ST(n)$, but in the discrete case can be applied to much more general manifolds. Furthermore it happens that in the classical $ST(n)$ case the continuous R–symmetry is an off–shell symmetry of the action while in the quantum $\widehat{ST}(n)$ case the discrete R–symmetry acts in general as an electric–magnetic duality rotation of the type of S–duality. As stated in section 2.4, the R-symmetry of rigid N=2 gauge theories should have a natural extension to the gravitationally coupled case. In principle, given a rigid supersymmetric theory, it is always possible to define its coupling to supergravity, yielding a locally supersymmetric theory. This does not mean that, starting with a complicated "dynamical" N=2 (or N=1) lagrangian, it is an easy task to define its gravitational extension. So we need some guidelines to relate the R-symmetry of a rigid theory to the R-symmetry of a corresponding locally supersymmetric theory. The main points to have in mind are the following ones:

- The R-symmetry group G_R, whether continuous or discrete, must act on the symplectic sections $(X, \partial F)$ by means of symplectic matrices:

$$\forall\, g \in G_R \;\; \hookrightarrow \;\; \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix} \in \Gamma_R \subset SP(4 + 2n, \mathbb{R}). \tag{4.1}$$

- The fields of the theory must have under G_R well defined charges, so that G_R is either a $U_R(1)$ group if continuous or a cyclic group $\mathbb{Z}_p$ if discrete.

- By definition the left–handed and right–handed gravitinos must have R– charges $q = \pm 1$, respectively

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- Under the G_R action there must be, in the special manifold, a preferred direction corresponding to the dilaton–axion multiplet whose R–charges are reversed with respect to those of all the other multiplets. As emphasized, this is necessary, in order for the topological twist to leave the axion–dilaton field physical in the topological theory, contrary to the other scalar partners of the vectors that become ghosts for the ghosts

The last point of the above list is an independent assumption from the previous three. In order to define a topological twist, the first three properties are sufficient and are guaranteed by N=2 supersymmetry any time the special manifold admits a symplectic isometry whose associated Kähler rescaling factor is $f_{2\vartheta}(z) = e^{2i\theta}$ (see below for more details). The third property characterizes the R–symmetry (or R–duality) of those N=2 supergravities that have an axion–dilaton vector multiplet.

For the classical coset manifolds $ST(n)$ the appropriate R-symmetry is continuous and it is easily singled out: it is the $SO(2) \sim U(1)$ subgroup of the isotropy group $SO(2) \times SO(n) \subset SO(2, n)$. The coordinates that diagonalize the R–charges are precisely the Calabi–Visentini coordinates discussed in the previous section. In the flat limit they can be identified with the special coordinates of rigid special geometry. Hence such gravitational R-symmetry is, as required, the supergravity counterpart of the R-symmetry considered in the rigid theories. Due to the direct product structure of this classical manifold the preferred direction corresponding to the dilaton–axion field is explicitly singled out in the $SU(1,1)/U(1)$ factor .

Generically, in the quantum case, the R-symmetry group G_R is discrete. Its action on the quantum counterpart of the Calabi–Visentini coordinates $\widehat{Y}^\alpha$ must approach the action of a discrete subgroup of the classical $U(1)_R$ in the same asymptotic region where the local geometry of the quantum manifold $\widehat{ST}(n)$ approaches that of $ST(n)$. This is the large radius limit if we think of $\widehat{ST}(n)$ as of the moduli–space of some dynamical Calabi–Yau threefold. To this effect recall that special Kähler geometry is the moduli–space geometry of Calabi–Yau threefolds and we can generically assume that any special manifold $\mathcal{SM}$ corresponds to some suitable threefold. Although the G_R group is, in this sense, a subgroup of the classical $U_R(1)$ group, yet we should not expect that it is realized by a subgroup of the symplectic matrices that realize $U(1)_R$ in the classical case. The different structure of the symplectic R–matrices is precisely what allows a dramatically different form of the special metric in the quantum and classical case. The need for this difference can be perceived *a priori* from the request that the quantum R-symmetry matrix should be symplectic integer valued. As we are going to see this is possible only for $\mathbb{Z}_4$ subgroups of $U(1)_R$ in the original symplectic embedding. Hence the different $\mathbb{Z}_p$ R–symmetries appearing in rigid quantum theories should have different symplectic embeddings in the gravitational case.

Let us now give the general properties of the gravitationally extended R-symmetry, postponing to section 4.2 the treatment of the specific case $ST(n)$.

4.1 The general form of R-symmetry in supergravity

R-symmetry is either a $U(1)$ symmetry or a discrete $\mathbb{Z}_p$ symmetry. Thus, if R-symmetry acts diagonally with charge q_R on a field ϕ, this means that $\phi \to e^{q_R i \vartheta} \phi$, $\vartheta \in [0, 2\pi]$ in the continuous case. In the discrete case only the values $\vartheta = \frac{2\pi}{p} l$, $l = 0, 1 \ldots p - 1$ are allowed and in particular the generator of the $\mathbb{Z}_p$ group acts as $\phi \to R\phi = e^{q_R \frac{2\pi i}{p}} \phi$.

By definition R-symmetry acts diagonally with charge $+1$ (-1) on the left-(right)-handed gravitinos (in the same way as it acts on the supersymmetry parameters in the rigid case):

$$
\begin{array}{ccc}
\psi_A \to e^{i\vartheta} \psi_A & & q_L(\psi_A) = 1 \\
\psi^A \to e^{-i\vartheta} \psi^A & \text{i.e.} & q_R(\psi^A) = -1.
\end{array}
\tag{4.2}
$$

R-symmetry generates isometries $z^i \to (R_{2\vartheta} z)^i$ [3] of the scalar metric g_{ij*} and it is embedded into $Sp(2n + 4, \mathbb{R})$ by means of a symplectic matrix:

$$
M_{2\vartheta} = \begin{pmatrix} a_{2\vartheta} & b_{2\vartheta} \\ c_{2\vartheta} & d_{2\vartheta} \end{pmatrix} \in Sp(2n + 4, \mathbb{R}).
\tag{4.3}
$$

As we have already pointed out it turns out that in the classical case of $ST(n)$ manifolds R–symmetry does not mix the Bianchi identities with the field equations since the matrix (4.1) happens to be block diagonal: $b_{2\theta} = c_{2\theta}$. In the quantum case, instead, this is in general not true. There is a symplectic action on the section (X^Λ, F_Λ) induced by $z^i \to (R_{2\vartheta} z)^i$:

$$
(X, F) \to f_{2\vartheta}(z^i) M_{2\vartheta} \cdot (X, F)
\tag{4.4}
$$

where the Kähler compensating factor $f_{2\vartheta}(z^i)$ depends in general both on the transformation parameter ϑ and on the base–point z. By definition this compensating factor is the same that appears in the transformation of the gravitino field $\psi_A \to \exp[f_{2\vartheta}(z^i)/2] \psi_A$. Since we have imposed that the transformation of the gravitino field should be as in (4.2) it follows that the R-symmetry transformation must be such as to satisfy eq.(4.4) with a suitable matrix (4.3) and with a compensating Kähler factor of the following specific form:

$$
f_{2\vartheta}(z^i) = e^{2i\vartheta}.
\tag{4.5}
$$

Condition (4.5) is a crucial constraint on the form of R-symmetry.
The action of the R-symmetry on the matrix $\mathcal{N}$ is determined by the form of the matrix $M_{2\vartheta}$ (see [36]):

$$
\mathcal{N} \to (c_{2\vartheta} + d_{2\vartheta}\mathcal{N})(a_{2\vartheta} + b_{2\vartheta}\mathcal{N})^{-1}
\tag{4.6}
$$

---

[3] As in the rigid case, the action of the R-symmetry group on the gravitinos, and more generally on the fermions, *doubly covers* its action on the bosonic fields. This property will become evident in eq. (4.5); it explains the chosen notation $(M_{2\vartheta})^i_l$ for the matrix expressing the R-action on the tangent bundle $T^{(1,0)}\mathcal{SM}$.

The supersymmetry transformation rules are encoded in the rheonomic parametrizations of the curvatures, summarized in Appendix C. For instance the supersymmetry transformations of the scalar fields are given by

$$\nabla z^i = \nabla_a z^i V^a + \overline{\lambda}^{iA}\psi_A \qquad \Rightarrow \qquad \delta_\epsilon z^i = \overline{\lambda}^{iA}\epsilon_A. \qquad (4.7)$$

Let us denote by J the Jacobian matrix of the transformations

$$(J_{2\vartheta})^i_l = \frac{\partial(R_{2\vartheta}z)^i}{\partial z^l}. \qquad (4.8)$$

If we now act on the scalars z^i by an R-transformation we conclude that, using eq.s (4.7,4.2)

$$\nabla z^i \to (J_{2\vartheta})^i_j \nabla z^j \qquad \Rightarrow \qquad \lambda^{iA} \to e^{-i\vartheta}(J_{2\vartheta})^i_j \lambda^{jA} \qquad (4.9)$$

Analogous considerations can be done for the hyperinos (cfr. eq. (C.13)).
• The supersymmetry transformation of the gravitino field are encoded in eq.s (C.16, C.17, C.23, C.24) and in their gauged counterparts (C.46, C.47). Requiring consistency with eq. (4.2) determines the R-charges of the various terms in the right hand side.

  i) The terms like $A^A_{B|b}$ that contain bilinear in the fermions are neutral (cfr. eq.s (C.31,C.32))

  ii) The R-symmetry acts diagonally on terms $T^\pm_{ab}$. These terms must have charge $q_R(T^\pm_{ab}) = \pm 2$. Notice that $T^\pm_{ab}$ can be expressed by the following symplectic invariants (see eq. (C.35))

$$T^-_{ab} \propto e^{\frac{K}{2}}(\overline{X}^\Lambda, \overline{F}_\Lambda) \cdot \begin{pmatrix} \widehat{\mathcal{F}}^{\Lambda-}_{ab} \\ \mathcal{N}_{\Lambda\Sigma}\widehat{\mathcal{F}}^{\Sigma-}_{ab} \end{pmatrix}, \qquad (4.10)$$

where $\widehat{\mathcal{F}}^{\Lambda-}_{ab} = \mathcal{F}^{\Lambda-}_{ab} + \frac{1}{8}\nabla_{i^*}\overline{f}^\Lambda_{j^*}\overline{\lambda}^{i^*}_A\gamma_{ab}\lambda^{j^*}_B\epsilon^{AB}$. Under an R transformation the symplectic product appearing in eq. (4.10) is left invariant up to the overall (antiholomorphic) factor coming from eq. (4.4), namely

$$T^-_{ab} \to \overline{f}_{2\vartheta}(\overline{z}^{i^*})T^-_{ab}. \qquad (4.11)$$

Since the R-symmetry act diagonally on T^-_{ab} and $q_R(T^-_{ab}) = -2$, we necessarily have

$$T^-_{ab} \to e^{-2i\vartheta}T^-_{ab}. \qquad (4.12)$$

Eq.s (4.11) and (4.12) are consistent with eq. (4.5).
• Let us consider the supersymmetry transformations of the gauginos, encoded in eq.s (C.27, C.28) [and their gauged counterpart (C.50,C.51)]. We impose that the Jacobian matrix is covariantly constant, $\nabla(J_{2\vartheta})^i_j = 0$. It then follows that the curvature $\nabla\lambda^{iA}$ transforms as λ^{iA}, that is as in eq. (4.9). We can in this way verify that the R transformations of $G^{-i^*}_{ab}$ (and its complex conjugate) transform consistently with the gaugino transformation.

The terms $\mathcal{Y}^{i^*}_{AB}$ are proportional to the Yukawa couplings C_{ijk}. These latter can be written in terms of a symplectic product:

$$C_{ijk} = (f_i, h_i) \cdot \nabla_j \begin{pmatrix} f_k \\ h_k \end{pmatrix}$$
$$(f_i^\Lambda, h_{\Lambda i}) \equiv e^{\mathcal{K}/2} \nabla_i (X^\Lambda, F_\Lambda). \tag{4.13}$$

Their R-transformation is therefore[4]:

$$C_{ijk} \rightarrow e^{4i\vartheta} (J_{2\vartheta}^{-1})_i{}^l (J_{2\vartheta}^{-1})_j{}^m (J_{2\vartheta}^{-1})_k{}^n C_{lmn}. \tag{4.14}$$

Utilizing eq. (4.14) in eq. (C.28) one can check that the transformation of $\mathcal{Y}^{i^*}_{AB} = g^{i^*j} C_{jlm} \overline{\lambda}^{lC} \lambda^{mD} \epsilon_{AC} \epsilon_{BD}$ is consistent with the transformation of the left hand side.

As can be easily verified, all the terms due to the gauging of the composite connections transform in the correct way to ensure the consistency of the R-transformations [see eq.s (C.45–C.54)].

- *Summarizing:*

The R-symmetry must act holomorphically on the scalar fields, $z^i \rightarrow (R_{2\vartheta} z)^i(z)$, being an isometry. Moreover the matrix $(J_{2\vartheta})^i_j$ has to be *covariantly constant*: $\nabla (J_{2\vartheta})^i_j = 0$. The R-transformation of parameter ϑ on the scalar fields must induce the transformation $(X, F) \rightarrow e^{2i\vartheta} M_{2\vartheta}(X, F)$, where $M_{2\vartheta}$ is of the form (4.3). In the topological twist, the ghost numbers are redefined as in eq. (2.10) by adding the R-charges.

- *The dilaton–axion direction in the discrete case:*

In the classical case of the $ST(n)$ manifolds the existence of a preferred direction is obvious from the definition of the manifolds and R–symmetry singles it out in the way discussed in the next section. Let us see how the dilaton–axion direction can be singled out by the discrete R–symmetry of the quantum manifolds $\widehat{ST}(n)$. Let $G_R = \mathbb{Z}_p$ and let $\alpha = e^{2\pi i/p}$ be a p–th root of the unity. In the space of the scalar fields z^i there always will be a coordinate basis $\{u^i\}$ ($i = 1, \ldots n+1$) that diagonalizes the action of $R_{2\vartheta}$ so that:

$$R_{2\vartheta} u^i = \alpha^{q_i} u^i \qquad q_i = 0, 1, \ldots, p-1 \bmod p \tag{4.15}$$

The $n+1$ integers q_i (defined modulo p) are the R–symmetry charges of the scalar fields u_i. At the same time a generic $Sp(4 + 2n, \mathbb{R})$ matrix has eigenvalues:

$$\left(\lambda_0, \lambda_1, \ldots, \lambda_{n+1}, \frac{1}{\lambda_0}, \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_{n+1}} \right) \tag{4.16}$$

The R–symmetry symplectic matrix $M_{2\vartheta}$ of eq. (4.3), being the generator of a cyclic group $\mathbb{Z}_p$, has eigenvalues:

$$\lambda_0 = \alpha^{k_0}, \quad \lambda_1 = \alpha^{k_1}, \quad \ldots, \quad \lambda_{n+1} = \alpha^{k_{n+1}}, \tag{4.17}$$

---

[4]Indeed the section (f_i, h_i) transforms into $e^{2i\vartheta} M_{2\vartheta}((J_{2\vartheta}^{-1})_i{}^l f_l, (J_{2\vartheta}^{-1})_i{}^l h_l)$. Then eq. (4.14) follows. Notice that this transformations is the appropriate one for a section of $\mathcal{L}^2_H \times [T^{(1,0)} \mathcal{SM}]^3$, that is the correct interpretation of the C_{ijk}'s.

where $(k_0, k_1, \ldots k_{n+1})$ is a new set of $n + 2$ integers defined modulo p. These numbers are the R–symmetry charges of the electric–magnetic field strengths

$$F_{\mu\nu}^0 + \mathrm{i}\, G_{\mu\nu}^0, \quad F_{\mu\nu}^1 + \mathrm{i}\, G_{\mu\nu}^1, \quad \ldots \quad F_{\mu\nu}^{n+1} + \mathrm{i}\, G_{\mu\nu}^{n+1}\,, \tag{4.18}$$

their negatives, as follows eq. (4.16), being the charges of the complex conjugate combinations $F_{\mu\nu} - \mathrm{i}G_{\mu\nu}$. Since what is really relevant in the topological twist are the differences of ghost numbers (not their absolute values), the interpretation of the scalars u^i $(i = 1, \ldots, n)$ as ghost for ghosts and of the corresponding vector fields as physical gauge fields requires that

$$q_i \;=\; k_i \,+\, 2 \qquad i = 1, \ldots, n \tag{4.19}$$

On the other hand, if the vector partner of the axion–dilaton field has to be a ghost for ghosts, the S–field itself being physical, we must have:

$$k_{n+1} = q_{n+1} + 2 \tag{4.20}$$

In eq. (4.20) we have conventionally identified

$$S = u^{n+1} \tag{4.21}$$

Finally the R–symmetry charge k_0 of the last vector field–strength $F_{\mu\nu}^0$ is determined by the already established transformation eq. (4.12) of the graviphoton combination (4.10)[5]

Summarizing, this situation is similar to that occurring in the topological Landau–Ginzburg models [37] where the physical scalar fields X^i have a non–zero R–symmetry charge equal to their homogeneity weight [54, 45] in the superpotential $\mathcal{W}(X)$. After topological twist they acquire a non zero (fractional) ghost–number that however differs from the ghost–number of the fermions by the correct integer amount. [6]

4.2 R–symmetry in the $ST(n)$ case

In the case of the microscopic lagrangian the special Kähler manifold of the scalars is a $ST(n)$ manifold. The action of R-symmetry is extremely simple. As already

---

[5] *[Note added in proofs]*

In [60] an explicit example is provided of quantum R–symmetry based on the local N=2 $SU(2)$ gauge theory associated with the Calabi–Yau manifold $WCP_4(8; 2, 2, 2, 1, 1)$ of Hodge numbers $(h_{11} = 2, h_{21} = 86)$ that has been considered by Vafa and Kachru in [61] as an example of heterotic/type II duality.

[6]*[Note added in proofs]* In a very recent paper [62] it has been proposed a new interpretation of the D=2 topological Landau–Ginzburg models, based on BRST, anti–BRST symmetry where, notwithstanding the fractional R–charges, the ghost numbers become integer. It would be interesting to inquiry whether such analysis can be extended to the case of D=4 vector multiplets

stated in section 2, see eq. (2.30), the S field has to be neutral, while the Y^α fields have R-charge 2:

$$\begin{cases} S \to S \\ Y^\alpha \to \mathrm{e}^{2\mathrm{i}\vartheta} Y^\alpha \end{cases} \quad \Rightarrow \quad (J_{2\vartheta})^i_j = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\vartheta} \delta^\alpha_\beta \end{pmatrix}. \tag{4.22}$$

Using the factorized form eq. (3.14) of the metric, it is immediate to check that the matrix $J_{2\vartheta}$ is covariantly constant.

Utilizing the explicit form eq. (3.12) of the symplectic section, eq. (4.22) induces the transformation:

$$\begin{pmatrix} X \\ F \end{pmatrix} \to \mathrm{e}^{2\mathrm{i}\vartheta} \begin{pmatrix} m_{2\vartheta} & 0 \\ 0 & (m^T_{2\vartheta})^{-1} \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix}$$

$$m_{2\vartheta} = \begin{pmatrix} \cos 2\vartheta & -\sin 2\vartheta & 0 \\ \sin 2\vartheta & \cos 2\vartheta & 0 \\ 0 & 0 & \mathbb{1}_{n \times n} \end{pmatrix} \in SO(2, n). \tag{4.23}$$

We see that the crucial condition (4.5) is met. Furthermore note that in this classical case $b_{2\vartheta} = c_{2\vartheta} = 0$, the matrix (4.3) is completely diagonal and it has the required eigenvalues $(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}, 1, \dots, 1)$.

At this point we need no more checks; the R-symmetry defined by eq. (4.22) is a true symmetry of the lagrangian and satisfies all the expected properties. The gauge fields A^α do not transform, while the A^0, A^S gauge fields undergo an $SO(2)$ rotation:

$$\begin{cases} \begin{pmatrix} A^0 \\ A^S \end{pmatrix} \to \begin{pmatrix} \cos 2\vartheta & -\sin 2\vartheta \\ sin 2\vartheta & \cos 2\vartheta \end{pmatrix} \begin{pmatrix} A^0 \\ A^S \end{pmatrix} \\ \\ A^\alpha \to A^\alpha. \end{cases} \tag{4.24}$$

Notice that from eq.s (4.23), (4.24) and from the explicit form of the embedding (3.11) we easily check that the R-symmetry for the $ST(n)$ case is nothing else but the $SO(2) \sim U(1)$ subgroup of the isometries appearing in the denominator of the coset $SO(2, n)/SO(2) \times SO(n)$.

At the quantum level the R–symmetries should act on the symplectic sections as a symplectic matrix belonging to $Sp(2n + 4, \mathbb{Z})$. Consider then the intersection of the continuous R symmetry of eq.s (4.22,4.23) with $Sp(2n + 4, \mathbb{Z})$: the result is a $\mathbb{Z}_4$ R-symmetry generated by the matrix $M_{2\vartheta}$ with $\vartheta = \pi/4$, where:

$$m_{\pi/2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{n \times n} \end{pmatrix} \in SO(2, n; \mathbb{Z}). \tag{4.25}$$

As already observed, in a generic case, after the quantum corrections are implemented, the discrete R-symmetry $\mathbb{Z}_p$ is a subgroup of $U(1)_R$ as far as the action on the moduli at large values is concerned, but it is implemented by $Sp(4 + 2n, \mathbb{Z})$

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Table 1: *Spin–R charges assignments*

Field	$SU(2)_L$	$SU(2)_R$	$SU(2)_I$	$SU(2)_Q$	R	gh'
V_a^μ	1/2	1/2	0	0	0	0
$\psi_{\mu A}$	1/2	0	1/2	0	1	1
ψ_μ^A	0	1/2	1/2	0	-1	-1
$A_\mu^0 + iA_\mu^S$	0	0	0	0	2	2
$A_\mu^0 - iA_\mu^S$	0	0	0	0	-2	-2
A_μ^α	0	0	0	0	0	0
S	0	0	0	0	0	0
y^α	0	0	0	0	2	2
$\overline{y}^\alpha$	0	0	0	0	-2	-2
λ^{SA}	1/2	0	1/2	0	-1	-1
λ_A^{S*}	0	1/2	1/2	0	1	1
$\lambda^{\alpha A}$	1/2	0	1/2	0	1	1
$\lambda_A^{\alpha*}$	0	1/2	1/2	0	-1	-1
$u^A{}_{\overline{A}}{}^t$	0	0	1/2	1/2	0	0
$\zeta^{\overline{A}t}$	1/2	0	0	1/2	-1	-1
$\zeta_{\overline{A}}{}^t$	0	1/2	0	1/2	1	1

matrices that are not the restriction to discrete value of theta of the matrix $M_{2\vartheta}$ defined in eq.s (4.23). In the one modulus case where, according to the analysis by Seiberg–Witten the rigid R-symmetry is $\mathbb{Z}_4$, there is the possibility of maintaining the classical form of the matrix $M_{2\vartheta}$ also at the quantum level and in the case of local supersymmetry. This seems to be a peculiarity of the one–modulus N=2 gauge theory.

To conclude, in table 1 we give the R-symmetry charge assignments for the fundamental fields of the $ST(n)$ case together with the spin and R-symmetry assignments for the hyperini and for quaternionic vielbein u, which will be properly defined in appendix A. Notice that in this table, concerning the quaternionic sector, we have explicitly splitted the $SO(4)$ index a (see appendix A for details) into the $SU(2)_I \times SU(2)_Q$ indices $(A, \overline{A})$ so that $u^{at} \equiv u^A{}_{\overline{A}}{}^t$. This splitting is fundamental, in order to redefine correctly the Lorentz group for the twist, so that, after the twist prescription is performed, the quaternionic vielbein become a Lorentz vector. This is consistent with the fact that u appear in the topological variation of $\zeta^{\overline{A}t}$, which acquires spin 1 after the twist. But we are going to analyse these problems in the following section.

5 The twist procedure

In this section we perform the topological twist–shift, following the four steps pointed out in the introduction.

Step i) is explicitly done following the procedure indicated in [44, 3, 4]. We extend the forms to ghost–forms, and we set

$$\widehat{d} = d + s \tag{5.1}$$

then we read the BRST variation of each field from the rheonomic parametrization displayed in Appendix C, selecting out the terms with the appropriate ghost numbers. This step is a purely algorithmic one, and we do not find convenient to write it in a fully extended form. A simplified example of this calculation will be presented analyzing step iii) and iv) of the twist–shift procedure, when we consider the variations of the (topological) antighosts. These variations are the only one we are ultimately interested, since they give the "instanton" conditions of our topological field theory. The second step is immediate. We have analyzed in section 4 the gravitational extended R-symmetry associated with all the fields of our model. This global symmetry is utilized to redefine the ghost number according to equation (2.10).

Let us now consider with more detail steps iii) and iv). The twist is obtained by redefining the Lorentz group as in eq.s (2.8,2.9). The spin assignments of the fundamental fields of our theory is resumed in table 1. Following the notations of references [4] we classify each field, before the twist, by the expression $^{r}(L, R, I, Q)^{g}_{f}$, where (L, R, I, Q) are the representation labels for $(SU(2)_L, SU(2)_R, SU(2)_I, SU(2)_Q)$, r is the R–charge assignments and f, g denote the ghost number and the form degree. The twist procedure is summarized as follows:

$$
\begin{aligned}
SU(2)_L &\rightarrow& SU(2)'_L &= \mathrm{diag}[SU(2)_L \otimes SU(2)_Q] \\
SU(2)_R &\rightarrow& SU(2)'_R &= \mathrm{diag}[SU(2)_R \otimes SU(2)_I] \\
U(1)_g &\rightarrow& U(1)'_g &= \mathrm{diag}[U(1)_g \otimes U(1)_R] \\
^{r}(L, R, I, Q)^{g}_{f} &\rightarrow& (L \otimes Q, I \otimes R)^{g+r}_{f}. &
\end{aligned} \tag{5.2}
$$

The second fundamental ingredient is the topological shift. As anticipated in section 2.2 this is a shift by a constant of the $(0,0)^0_0$–field coming by applying the twist algorithm to the right handed components of the supersymmetry ghost. Let us denote this ghost by c^A, with spinorial components $c^{\dot{\alpha} A}$. As it is immediately verified c^A has the following quantum numbers, before the twist:

$$^{-1}(0, 1/2, 1/2, 0)^1_0. \tag{5.3}$$

According to the prescription (5.2) we identify the $SU(2)_R$ index $\dot{\alpha}$ with the $SU(2)_I$ index A, and we perform the shift by writing

$$c^{\dot{\alpha} A} \rightarrow -\frac{i}{2} e \epsilon^{\dot{\alpha} A} + c^{\dot{\alpha} A}. \tag{5.4}$$

In eq. (5.4), e is the "broker". The broker, as introduced in ref. [4], is a zero–form with fermion number one and ghost number one. It is a formal object which rearranges the form number, ghost number and statistic in the correct way and it appears only in the intermediate steps of the twist. e^2 has even fermion number and even ghost number, and can be normalized to $e^2 = 1$.

The BRST quantized topological field theory is thus defined by the new set of fields, obtained from the untwisted ones by changing the spins and the ghost numbers; and by the shifted BRST charge, which is the sum of the old one plus the shifted component of the supersymmetry charge. In our approach we are not interested in writing down all the twisted–shifted variation. We just point our attention to the variations of the (topological) antighosts, namely the fields $\psi^A, \lambda^{SA}, \lambda_A^{\alpha^\star}, \zeta^{\overline{A}t}$ appearing in table 1. Such variations (or some particular projections of these variations) will define the instantons of our theory. As anticipated, we are looking for $(0,0)_0^0$ component of the supersymmetry ghost $c^{\dot{\alpha}A}$. Moreover, to select the instanton conditions we set to zero all the fields which have non zero ghost number.

Let us firstly consider the variation of the right handed gravitino $\psi^{\dot{\alpha}A}$. Following equation (5.2) we find that

$$\psi^A \leftrightarrow {}^{-1}(0,1/2,1/2,0)_1^0 \rightarrow (0,0)_1^{-1} \oplus (0,1)_1^{-1}. \tag{5.5}$$

As a consequence, in the "extended" ghost–form $\psi^A = \psi^A + c^A$, the supersymmetry ghost c^A, which has labels as described in eq. (5.3), becomes, after the twist:

$$c^A \leftrightarrow {}^{-1}(0,1/2,1/2,0)_0^1 \rightarrow (0,0)_0^0 \oplus (0,1)_0^0. \tag{5.6}$$

To read off the gravitational instanton condition we have just to consider the variation of the gravitino along the $(0,0)_0^0$ component of c^A, and to set to zero all the non physical fields.

Actually, we better consider the gravitino with the field redefinition $\psi^A \rightarrow e^{\frac{K}{4}}\psi^A$, in such a way that, in the curvature definition, only the holomorphic component of the Kähler connection appears. Moreover, in presence of gauging, the Kähler and the $SU(2)_I$ quaternionic connections are extended as in Appendix C, i.e.

$$\begin{aligned} \widehat{Q} &= Q + gA^\Lambda \mathcal{P}_\Lambda^0 \\ \widehat{\omega}^{-x} &= \omega^{-x} + gA^\Lambda \mathcal{P}_\Lambda^{-x}. \end{aligned} \tag{5.7}$$

It is quite immediate to verify that $\mathcal{P}_\Lambda^0$ does not give any contribution to the variation of ψ^A (at ghost number zero), while the only contribution to $\widehat{\omega}^{-x}$ come from the $SO(n)$ indices, i.e.

$$\widehat{\omega}^{-x} = \omega^{-x} + gA^\alpha \mathcal{P}_\alpha^{-x} \tag{5.8}$$

The twist procedure permits the following identification $\psi^{\dot{\alpha}A} \rightarrow \psi^{\dot{\alpha}\dot{A}}$, where we identify the left handed Lorentz index $\dot{\alpha}$ with the $SU(2)_I$ one $A = \dot{A}$. Next, we define the following fields (see reference [3]):

$$\begin{aligned} \widetilde{\psi}^{ab} &= -e\overline{\sigma}_{\dot{\alpha}\dot{A}}^{ab}\psi^{\dot{\alpha}\dot{C}}\epsilon_{\dot{C}\dot{A}} \end{aligned} \tag{5.9}$$

$$\begin{aligned} \widetilde{\psi} &= -e\psi^{\dot{\alpha}\dot{C}}\epsilon_{\dot{C}\dot{A}}\delta_{\dot{A}}^{\dot{\alpha}} \end{aligned} \tag{5.10}$$

where $\overline{\sigma}^{ab}$ are defined in appendix C [actually here we use the euclidean version of the matrices defined in (C.10)]. Looking at the curvature definition (C.17) and at the rheonomic parametrization (C.24) we find that the only contributions coming from the ghost zero sector along the (shifted part) of the supersymmetry ghost are:

$$\delta\widetilde{\psi}^{ab} \;=\; \frac{i}{2}(\omega^{-ab} - \sum_{u=1}^{3} I_u^{ab}\widehat{\omega}^{-u}) \tag{5.11}$$

$$\delta\widetilde{\psi} \;=\; \frac{i}{2}Q_{\text{hol}}(S) \tag{5.12}$$

where the matrices $I_u^{ab} = -\frac{i}{2}Tr(\overline{\sigma}^{ab}\sigma_u^T)$, $u = 1,2,3 \equiv x,y,z$ can be identified (up to a trivial SO(3) rotation) with the anti-selfdual matrices J_u^{-ab} introduced in (A.24,A.25). Eq. (5.11) becomes precisely the first of eq.s (2.31), once expressed in terms of the curvatures.

Moreover, in eq. (5.12), Q_{hol} is given by

$$Q_{\text{Hol}} = -\frac{1}{4}\partial_S \mathcal{K} V_\mu^a \partial_a S. \tag{5.13}$$

Therefore the instanton condition $\delta\widetilde{\psi} = 0$ corresponds, in the euclidean formalism, to the Rey instantons. Indeed

$$\partial_a S = 0 \qquad \Leftrightarrow \qquad \partial_a D = \epsilon_{abcd}e^D H^{bcd}. \tag{5.14}$$

Let us go on and consider the instanton condition obtained from the variation of the gaugino λ^{SA}. In this case there is just a term which contribute, namely

$$\delta\lambda^{SA} = i\partial_a S\gamma^a c^A \tag{5.15}$$

so that the instanton condition obtained from eq. (5.15) is the same as the one obtained from eq. (5.12).

Working in a similar way on the antighost $\lambda_A^{\alpha^*}$ and using the formulæ for the metric tensor, for $G_{ab}^{-\alpha^*}$, $Y_{AB}^{\alpha^*}$ and for $W_{AB}^{\alpha^*}$, given in appendix C, we find the following condition

$$\mathcal{F}^{-\alpha\,ab} = \frac{g}{2\exp D}J_u^{-ab}\mathcal{P}_\alpha^{-u}. \tag{5.16}$$

Notice that eq. (5.16) identify the anti self dual part of the gauge connections with the quaternionic momentum map $\mathcal{P}^{-u}$ times the square of the *effective gauge coupling*. Indeed by performing the redefinition $A^\alpha \to \frac{1}{g}A^\alpha$ we precisely get

$$\mathcal{F}^{-\alpha^*\,ab} = \frac{1}{2}g_{eff.}^2 J_u^{-ab}\mathcal{P}_{\alpha^*}^{-u} \tag{5.17}$$

with $g_{eff.} = \frac{g}{\sqrt{\exp D}}$.

Finally, the instanton condition arising from the topological variation of the hyperini $\zeta_\alpha^{\overline{A}t}$ gives the following equations:

$$
\begin{aligned}
V^{\mu[a}u_I^{b]t}\nabla_\mu q^I &= 0 \\
V_a^\mu u_I^{at}\nabla_\mu q^I &= 0
\end{aligned}
\tag{5.18}
$$

where u^{at} is the vielbein defined in eq. (A.21). Eq.s (5.18) define the so called "gauged triholomorphic maps". To rewrite them in the more compact notation appearing in eq. (2.31) we have to define the three almost quaternionic structures in $M_{\text{space−time}}$ and $HQ(m)$, namely

$$
\begin{aligned}
(j_u)_\mu^\nu &\equiv J_u^{-ab}V_{\mu a}V_b^\nu \\
(J_u)_J^I &\equiv (J_u^-)_a{}^b u_J^{at}u_{bt}^I.
\end{aligned}
\tag{5.19}
$$

Using eq.s (5.19) we can easily rewrite eq.s (5.18) as in eq. (2.31) [4]

6 Dual description of the effective theory of N=2 heterotic string

In rigid N=2 supersymmetry, in order to describe the strong coupling regime of a non–abelian gauge theory of a group $\mathcal{G}$ it is useful to consider the dual effective theory which is also an N=2 gauge theory with the following differences:

i) The new gauge group $\widetilde{\mathcal{H}} = U(1)^r$ is abelian.

ii) The self–interaction of the abelian gauge multiplets is encoded in a non flat special geometry possessing a discrete group of duality symmetries.

iii) In addition to the gauge multiplet the dual theory contains a certain number of hypermultiplets that represent the monopoles of the original theory. This means that $\widetilde{\mathcal{H}}$ is actually the dual of the maximal torus $\mathcal{H} \subset \mathcal{G}$ of the original gauge group.

When the rigid Yang–Mills theory is embedded in a supersymmetric theory arising as a low energy limit of heterotic superstring, it is natural to associate to it a Calabi–Yau threefold [42] and a dual theory, which is a type II string theory [55, 56, 57, 58] compactified on that particular manifold. If we consider in this dual frame eq.s (2.31) we see that the existence of a non–trivial monopole background of a $U(1)$ field requires, in order to be consistent with N=2 Susy, the existence of background hypermultiplets that are charged with respect to the Ramond–Ramond $U(1)$ gauge fields. Since these hypermultiplets carry a Ramond charge they must appear as solitonic excitations of a type II string propagating on the C.Y. manifold. Evidence for the existence of such states has been given recently in [40] by studying the behaviour of the periods around the vanishing cycles of the Calabi–Yau manifold.

Appendix A: Structure and parametrization of the $HQ(m)$ quaternionic manifolds

It is possible to describe the $SO(4, m)/SO(4) \times SO(m)$ manifold as a "quaternionic quotient" of the (quaternionic) projective plane $\mathbb{HP}^{4(m+3)}$ with respect to an $SU(2)$ action. Such a description allows an explicit parametrization of the manifold in terms of a set of quaternionic coordinates. In the following we give such a parametrization together with some properties of quaternionic manifolds. We have no claim to mathematical completeness, and we refer the reader to [59] for more details on the subject

First of all, we realize the quaternionic units e_x, $x = 1, 2, 3$, satisfying the quaternionic algebra

$$e_x e_y = -\delta_{xy} + \epsilon_{xyz} e_z \tag{A.1}$$

by means of 2×2 matrices, setting $e_x \equiv -i\sigma_x$. By σ_x we denote the standard Pauli matrices. The e_x are imaginary units since $\overline{e}_x \equiv e_x^\dagger = -e_x$. It will be convenient to treat also the unit matrix on the same footing, setting $e_0 \equiv \mathbb{1}$ and thus having $\{e_a\} \equiv \{\mathbb{1}, -i\sigma_x\}$, $a = 0, 1, 2, 3$. Then it is immediate to write the one-to-one correspondence between points $\{x^a\}$ in $\mathbb{R}^4$ and quaternions q by setting

$$q = x^a e_a = \begin{pmatrix} u & i\overline{v} \\ iv & \overline{u} \end{pmatrix}, \qquad \overline{q} = x^a \overline{e}_a = \begin{pmatrix} \overline{u} & -i\overline{v} \\ -iv & u \end{pmatrix} \tag{A.2}$$

where $u = x^0 - ix^3$ and $v = -(x^1 + ix^2)$. The quaternionic projective space $\mathbb{HP}^{4(m+3)}$ can be described by the set of quaternions $\{q^I\}$, $I = 0, 1, \ldots m + 3$ satisfying

$$\begin{cases} \overline{q}^I q^J \eta_{IJ} = \mathbb{1} & \text{where} \quad \eta_{IJ} = \text{diag}(1, 1, 1, 1, -1, -1, \ldots) \\ \\ \{q^I\} \sim \{q^I \, \nu\} & \text{with} \quad \overline{\nu}\nu = \mathbb{1} \end{cases} \tag{A.3}$$

In eq. (A.3) the unit quaternion ν is, in our 2×2 realization, a SU(2) matrix.

The above description is the analogue of the usual description of a $\mathbb{CP}^N$ space, where the role of the SU(2) element ν is played by a phase, i.e. an element of U(1). Notice, however that the quaternionic product is non-commutative and the choice of ν acting from the right in eq. (A.3) is relevant.

The fundamental quaternionic one-form gauging this right SU(2) action is

$$\omega^- = \overline{q}^I dq_I. \tag{A.4}$$

The index are contracted with η_{IJ}; the choice of the notation ω^- for the $SU(2)$ connection will be clear in the sequel. Its curvature, defined as $\Omega^- = d\omega^- - \omega^- \wedge \omega^-$, is

$$\Omega^- = d\overline{q}^I \wedge dq_I - \overline{q}^I dq_I \wedge \overline{q}^J dq_J. \tag{A.5}$$

It is immediate to verify that Ω^- is covariantly closed. This 2-form is the quaternionic analogue of the Kähler form of $\mathbb{CP}^N$. Indeed, writing $\Omega^- = \frac{1}{2} \sum_{x=1}^{3} \Omega^{-\,x} e_x^T$, we have that $\Omega^{-\,3}$ is the Kähler form, the metric being

$$\mathrm{d}s^2 \mathbb{1} = \mathrm{d}\overline{q}^I \otimes \mathrm{d}q_I - \overline{q}^I \mathrm{d}q_I \otimes \overline{q}^J \mathrm{d}q_J \tag{A.6}$$

Consider now the *left* action of an $SU(2)$ on $\mathbb{HP}^{4(m+3)}$: $q^I \to \mu q^I$, with $\overline{\mu}\mu = \mathbb{1}$. The infinitesimal action is

$$\delta_x q^I = e_x q^I \tag{A.7}$$

Such transformations leave the metric invariant, and they leave the quaternionic structure invariant up to a gauge transformation. This property can be reexpressed as

$$\mathbf{i}_x \Omega^- = -\nabla \mathcal{P}_x^-, \qquad \text{where} \qquad \mathcal{P}_x^- \propto \overline{q}^I e_x q_I \tag{A.8}$$

where $\mathbf{i}_x$ denote the contraction along the killing vector in the x direction, $\mathbf{k}_x = e_x \frac{\partial}{\partial q^I} - \frac{\partial}{\partial \overline{q}^I} e_x$.

The quaternionic functions $\mathcal{P}_x^-$ are the quaternionic momentum map functions for the left $SU(2)$ action. They are the key ingredient needed to perform the quaternionic reduction of $\mathbb{HP}^{4(m+3)}$ with respect to this action. The quaternionic reduction procedure consists in the following two steps.

1. Restriction to the null level set of the momentum map,

$$\bigcap_x (\mathcal{P}_x^-)^{-1}(0). \tag{A.9}$$

   The dimension of the level set surface is $\dim\mathbb{HP}^{4(m+3)} - 3 \times 3$ as for every quaternion $\mathcal{P}_x^-$ $x = 1, 2, 3$ three real conditions are imposed. The level set surface can be shown to be invariant with respect to the action of the group for which $\mathcal{P}_x^-$ are the momentum map functions.

2. Quotient of the level-set surface eq. (A.9) with respect to the action of the group itself (in this case the left action of $SU(2)$, eq. (A.7)).

The dimension of the resulting quotient manifold, which is usually denoted as $\mathbb{HP}^{4(m+3)}//SU(2)$, is the dimension of the level set minus the dimension of $SU(2)$, that is

$$\dim\mathbb{HP}^{4(m+3)}//SU(2) = \dim\mathbb{HP}^{4(m+3)} - 3 \times 3 - 3 = 4m; \tag{A.10}$$

By the general properties of the quaternionic reduction, the quotient manifold is quaternionic, when it is equipped with the quaternionic structure obtained by restricting that of $\mathbb{HP}^{4(m+3)}$ to the level set (eq. (A.9)) and projecting it to the quotient. The quaternionic quotient construction implies that we can describe $\mathbb{HP}^{4(m+3)}//SU(2)$ by parametrizing a set of $4(m + 4)$ quaternions q^I, $I =$

$0, \ldots, m+3$ in terms of $4m$ independent real variables, so that the following equations holds:

$$\begin{cases} \overline{q}^I q_I = 1 \\ \overline{q}^I e_x q_I = 0 \qquad \forall x = 1, 2, 3 \end{cases}. \tag{A.11}$$

The first equation comes from the definition of the $\mathbb{HP}^{4(m+3)}$ space, (eq. (A.3)), the other equations define the level set of the $\mathcal{P}_x^-$ functions. We need to fix the gauge for the left SU(2) acting as $q^I \to \mu q^I$, but we also have to recall that the coordinates q^I were defined up to an SU(2) acting on the right: $q^I \to q^I \nu$, with $\overline{\nu}\nu = \overline{\mu}\mu = \mathbb{1}$.

Let us use the following notation:

$$q^I = \begin{pmatrix} U^I & iV^I \\ i\overline{V}^I & \overline{U}^I \end{pmatrix}. \tag{A.12}$$

We split the index $I = 0, 1, \ldots, m+3$ into $a = 0, 1, 2, 3$ and $t = 4, 5, \ldots, m+3$. We choose the quaternions

$$q^t = \begin{pmatrix} u^t & iv^t \\ i\overline{v}^t & \overline{u}^t \end{pmatrix}. \tag{A.13}$$

to represent the independent 4m real coordinates. In terms of the U^I, V^I, the equations (A.11) become

$$\begin{cases} U^I U_I = 0 \\ \overline{U}^I U_I = 1/2 \end{cases} \qquad \begin{cases} V^I V_I = 0 \\ \overline{V}^I V_I = 1/2 \end{cases} \qquad \begin{cases} U^I V_I = 0 \\ \overline{U}^I V_I = 0 \end{cases} \tag{A.14}$$

Notice that for $V^I = 0$ (and with I assuming only $m+2$ values) these equations reduce to the equations defining $SO(2, m)/SO(2) \times SO(m)$, in terms of the Calabi-Visentini coordinates $U^I \equiv Y^I$, and viceversa. Therefore we expect the solution to the complete set of equations to be similar to a pair of Calabi-Visentini systems suitably coupled.

Let us denote by $u^2, u \cdot v, \ldots$ the scalar products (SO(m) invariants) $u^t u^s \delta_{ts}$, $u^t v^s \delta_{ts}$, $\ldots$. A solution to eq.s (A.14) is

$$U = \frac{1}{\mathcal{N}_U(u, v)} \begin{pmatrix} 1/2(1 + u^2) \\ i/2(1 - u^2) \\ A(u, v) \\ -iA(u, v) \\ u^s \end{pmatrix} \qquad V = \frac{1}{\mathcal{N}_V(u, v)} \begin{pmatrix} B(u, v) \\ +iB(u, v) \\ 1/2(1 + v^2) \\ i/2(1 - v^2) \\ v^s \end{pmatrix} \tag{A.15}$$

where

$$\begin{cases} A(u, v) = \frac{1}{1 - |u^2 v^2|^2} \left[u \cdot v - u^2 \overline{u} \cdot v + u^2 v^2 (\overline{u} \cdot \overline{v} - \overline{u}^2 u \cdot \overline{v}) \right] \\ \\ B(u, v) = \frac{1}{1 - |u^2 v^2|^2} \left[\overline{u} \cdot v - v^2 \overline{u} \cdot \overline{v} + \overline{u}^2 v^2 (u \cdot \overline{v} - \overline{v}^2 u \cdot v) \right] \end{cases} \tag{A.16}$$

and where $\mathcal{N}_U(u, v), \mathcal{N}_V(u, v)$ are two normalization constant satisfying $\mathcal{N}_V(u, v) = \mathcal{N}_U(v, \overline{u})$, which are determined using the second row in the constraints (A.14). Notice that the V^I are obtained from the U^I by substituting $u \to v, v \to \overline{u}$.

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The quaternionic structure and the metric of $\mathbb{IHP}^{4(m+3)}$, eq.s (A.4,A.5,A.6) for the quotient manifold $\mathbb{IHP}^{4(m+3)}//SU(2)$ are obtained by substituting the explicit parametrization of eq.s (A.15,A.16) for the quaternions q^I. For instance, the connection for the right $SU(2)$ action becomes

$$\omega^- = \overline{q}^I(u,v)\mathrm{d}q_I(u,v) = \overline{q}^a(u,v)\mathrm{d}q^a(u,v) - \overline{q}(u,v)\cdot\mathrm{d}q(u,v) \tag{A.17}$$

• *Biquaternionic structure*

From now on we refer to $\mathbb{IHP}^{4(m+3)}//SU(2)$ and when we write q^I we mean $q^I(u,v)$. Beside the *right* $SU(2)$ action pertinent to the definition of $\mathbb{IHP}^{4(m+3)}$, in taking the quaternionic quotient we have introduced into the game a *left* $SU(2)$ action. Both these actions are gauged by a connection 1-form, from which a curvature 2-form is defined. This pair of curvature 2–forms constitutes *a pair of independent quaternionic structures* on $\mathbb{IHP}^{4(m+3)}//SU(2)$ that correspond to the same metric. The metric is left invariant by *both* $SU(2)$ actions and this restricts the holonomy group to $SU(2)\times SU(2)\times SO(m)$. We name quaternionic manifolds with such a reduced holonomy as biquaternionic manifolds. Here we just summarize our result for $\mathbb{IHP}^{4(m+3)}//SU(2)$

	Connection	Curvature	Metric
right SU(2)	$\omega^- = \overline{q}^I\mathrm{d}q_I$	$\Omega^- \equiv \mathrm{d}\omega^- - \omega^-\wedge\omega^-$ $= \mathrm{d}\overline{q}^I\wedge\mathrm{d}q_I - \overline{q}^I\mathrm{d}q_I\wedge\overline{q}^J\mathrm{d}q_J$	$\mathrm{d}s^2\mathbb{1} = \mathrm{d}\overline{q}^I\otimes\mathrm{d}q_I-$ $-\overline{q}^I\mathrm{d}q_I\otimes\overline{q}^J\mathrm{d}q_J$
left SU(2)	$\omega^+ = \mathrm{d}q^I\overline{q}_I$	$\Omega^+ \equiv \mathrm{d}\omega^+ - \omega^+\wedge\omega^+$ $= \mathrm{d}q^I\wedge\mathrm{d}\overline{q}_I - \mathrm{d}q^I\overline{q}_I\wedge\mathrm{d}q^J\overline{q}_J$	

$$\tag{A.18}$$

The "gauge" $SU(2)$ groups act as follows:

right SU(2)	left SU(2)
$q^I \to q^I\nu$	$q^I \to \mu q^I$
$\omega^- \to \overline{\nu}\omega^-\nu + \overline{\nu}\mathrm{d}\nu$	$\omega^- \to \omega^-$
$\omega^+ \to \omega^+$	$\omega^+ \to \mu\omega^+\overline{\mu} + \mathrm{d}\mu\overline{\mu}$
$\mathrm{d}s^2 \to \mathrm{d}s^2$	$\mathrm{d}s^2 \to \mathrm{d}s^2$

$$\tag{A.19}$$

• *The coset space $SO(4,m)/SO(4)\times SO(m)$*

A $SO(4,m)$ matrix $L^I{}_J$ satisfies

$$L^T\eta L = \eta \qquad \text{i.e.} \qquad L^I{}_K L^J{}_M\eta_{IJ} = \eta_{KM} \tag{A.20}$$

The left-invariant 1-form $u = L^{-1}\mathrm{d}L$ satisfies the Maurer-Cartan equation $\mathrm{d}u + u\wedge u = 0$, that encodes the structure constants of the algebra. Let now L be an element

of the quotient $SO(4,m)/SO(4) \times SO(m)$, then the 1-form u can be interpreted in the following way

$$u = \begin{pmatrix} u^{ab} & u^{at} \\ u^{ta} & u^{st} \end{pmatrix} \qquad \begin{cases} u^{ab} & \text{SO(4) connection} \\ u^{at} & \text{Vielbein on the coset} \\ u^{st} & \text{SO(m) connection.} \end{cases} \qquad (A.21)$$

Moreover the Maurer-Cartan equation can be accordingly splitted in three equations:

$$\begin{cases} \mathrm{d}u^{at} + u^{ab} \wedge u^{bt} - u^{ts} \wedge u^{as} = 0 & \text{Torsion equation} \\ \mathrm{d}u^{ab} + u^{ac} \wedge u^{cb} = -u^{as} \wedge u^{bs} & \text{SO(4) curvature} \\ \mathrm{d}u^{ts} - u^{tr} \wedge u^{rs} = u^{at} \wedge u^{as} = 0 & \text{SO(m) curvature} \end{cases} \qquad (A.22)$$

The above equations describe the geometry of the coset space $\frac{SO(4,m)}{SO(4) \times SO(m)}$ in terms of coset representatives. Notice that the vielbein $u^{at} = u_I^{at} dq^I$ explicitly carries a vector index $a = 0, 1, 2, 3$ of SO(4) and an index t in the vector representation of SO(m), which means that the holonomy group is $SO(4) \times SO(m)$.

$\bullet$ *Identification of* $\mathbb{HP}^{4(m+3)}//SU(2)$ *with* $SO(4,m)/SO(4) \times SO(m)$

In the above notation the identification is provided by the position

$$q^I = \frac{1}{2} L^I{}_a e_a. \qquad (A.23)$$

With this position, one can easily check that the constraints eq. (A.11) turn into the orthogonality condition $L^I{}_a L^J{}_b \eta_{IJ} = \delta_{ab}$.

In eq. (A.23) we have converted SO(4) vectors into quaternions, that is object transforming in the fundamental of SU(2)×SU(2), by contracting them with the imaginary units $\{e_a\}$. To show the equivalence at the level of the connections and curvatures we must convert the adjoint indices of SO(4) into adjoint indices of SU(2)×SU(2). This conversion is realized by two set of 4×4 antisymmetric matrices $\{J^{+x}\}, \{J^{-x}\}$, $x = 1, 2, 3$, satisfying ($\epsilon_{0123} = 1$)

$$\begin{aligned} J^{\pm x} J^{\pm y} &= -\delta_{xy} + \epsilon_{xyz} J^{\pm z} \\ J^{\pm x}_{ab} &= \pm \frac{1}{2} \epsilon_{abcd} J^{\pm x}_{cd} \\ [J^{\pm x}, J^{\mp y}] &= 0 \quad \forall x, y. \end{aligned} \qquad (A.24)$$

They can be expressed in terms of the quaternionic units by the following key relation:

$$\begin{cases} J^{+x}_{ab} = 1/2 \, \mathrm{Tr}(e_a \bar{e}_b e_x^T) \\ J^{-x}_{ab} = -1/2 \, \mathrm{Tr}(e_a e_x^T \bar{e}_b) \end{cases} \qquad (A.25)$$

The identification between the SO(4) connection μ^{ab} of $SO(4,m)/SO(4) \times SO(m)$ and the SU(2)×SU(2) connections $\omega^{\pm}$ goes as follows. Set

$$\omega^{\pm} = \frac{1}{2} \omega^{\pm x} e_x^T. \qquad (A.26)$$

Then

$$u^{ab} = \frac{1}{2}(J_x^{+\,ab}\omega^{+\,x} + J_x^{-\,ab}\omega^{-\,x}) \quad \Leftrightarrow \quad \begin{cases} \omega^{+\,x} = \frac{1}{2}J_{ab}^{+\,x}u^{ab} \\ \\ \omega^{-\,x} = \frac{1}{2}J_{ab}^{-\,x}u^{ab} \end{cases} . \tag{A.27}$$

This can be checked substituting into the explicit expressions (A.18) of $\omega^{\pm}$ the identification (A.23) of the quaternions q^I.

At the level of curvatures we analogously set

$$\Omega^{\pm} = \tfrac{1}{2}\Omega^{\pm\,x}e_x, \tag{A.28}$$

and, recalling that by eq. (A.22) the SO(4) curvature is $-u^{as} \wedge u^{bs}$, we have

$$u^{as} \wedge u^{bs} = -\frac{1}{2}(J_x^{+\,ab}\Omega^{+\,x} + J_x^{-\,ab}\Omega^{-\,x}) \quad \Leftrightarrow \quad \begin{cases} \Omega^{+\,x} = -\frac{1}{2}J_{ab}^{+\,x}u^{as} \wedge u^{bs} \\ \\ \Omega^{-\,x} = -\frac{1}{2}J_{ab}^{-\,x}u^{as} \wedge u^{bs} \end{cases} . \tag{A.29}$$

Note that upon use of the definitions (A.26,A.28) the curvature definition $\Omega^{\pm} = \mathrm{d}\omega^{\pm} - \omega^{\pm} \wedge \omega^{\pm}$ is rewritten as $\Omega^{\pm\,x} = \mathrm{d}\omega^{\pm\,x} + \frac{1}{2}\epsilon_{xyz}\omega^{\pm\,y} \wedge \omega^{\pm\,z}$.

Appendix B: A note on Q–symmetry

In order to redefine the Lorentz group for the twist, we have to write the quaternionic vielbein as a doublet under both the $SU(2)_I$ and $SU(2)_Q$ groups, and as a vector under $SO(m)$. The group $SU(2)_Q$ for the classical manifolds is the normaliser of $SO(m)$ in the $Sp(2m)$ subgroup of the $Hol(\mathcal{QM}_{4m})$. Now, in those quantum cases, where the hypermultiplet metric receives corrections (type II string, for instance) it suffices that only a discrete subgroup of $SU(2)_Q$ survives, namely it is not necessary for the vierbein to be a doublet under a full $SU(2)_Q$ group. It is sufficient that it is a doublet under the isometries generated by a Kleinian finite group G_Q, whose normalizer in the holonomy group should be $SO(m)$. We name such group the Q–symmetry group. An interesting example is provided by the case where for G_Q we take the binary extension of the dihedral group D_2. In this example the vielbein is acted on by a second set of quaternionic structures (such as the J_u^+ we have defined for the classical case) acting on the index $\overline{A}$ in the fundamental representation of $SU(2)$. This means that the Q–symmetry group is composed of eight elements, namely the second set of quaternionic structures J_x^+, J_y^+, J_z^+, their opposite $-J_x^+, -J_y^+, -J_z^+$ and the two matrices ± 1. This, however, is just one possibility. In the same way as any cyclic group $\mathbb{Z}_p$ can emerge as R-symmetry group of the quantum special manifold, in the same way any Kleinian subgroup of $SU(2)$ can emerge as Q–symmetry of the quantum quaternionic manifold.

Appendix C: Rheonomic parametrizations of N=2 matter coupled supergravity

In this appendix we write the full set of rheonomic parametrization for the matter coupled N=2 supergravity pertaining the examples studied in this paper. These are essential ingredients while studying the topological variation of the fields, and we include them for completeness. Here we limit our exposition only to the essential points and to the formulae that are needed in the present paper. For a detailed treatment on this subject we refer to [25]. To write the set of curvature definitions and rheonomic parametrization we need to recall a procedure named in [25] "gauging of the composite connection". On the scalar manifold $ST(n) \times HQ(m)$ we can introduce several connection 1-forms related to different bundles. In particular we have the standard Levi–Civita connection and the $SU(2) \times U(1)$ connection $(\omega^-, \mathcal{Q})$, as defined in (A.5) and (3.16). Gauging the corresponding supergravity theory is done by gauging these composite connections in the underlying σ-model. For a Kähler manifold, if we call z^i the scalar fields[7] and $k^i(z)$ the Killing vectors, we have to replace the ordinary differential by the covariant ones:

$$dz^i \rightarrow \nabla z^i = dz^i + gA^\Lambda k_\Lambda(z) \tag{C.1}$$

together with their complex conjugate. In eq. (C.1) A^Λ is the gauge one form ($\Lambda = 0, S, \alpha$ in our case). At the same time the Levi–Civita connection $\Gamma^i_j = \Gamma^i_{jk} dz^k$ is replaced by:

$$\Gamma^i_j \rightarrow \widehat{\Gamma}^i_j \equiv \Gamma^i_{jk} \nabla z^k + gA^\Lambda \partial_j k^i_\Lambda \tag{C.2}$$

so that the curvature two form become (as in the previous equations we omit the obvious complex conjugate expression)

$$\widehat{R}^i_j = R^i_{jkl} \nabla z^k \wedge \nabla z^l + g\mathcal{F}^\Lambda \partial_j k^i_\Lambda \tag{C.3}$$

where $\mathcal{F}$ is the field strength associated with A^Λ. In a fully analogous way we can gauge the $Sp(2m)$ connection of the quaternionic scalar manifold, but we will now focus our attention on the $SU(2) \times U(1)$ connection. In this case the existence of the Killing vector prepotentials $\mathcal{P}^0_\Lambda, \mathcal{P}^{-x}_\Lambda$ ($x = 1, 2, 3$) permits the following covariant definitions:

$$\begin{aligned} \mathcal{Q} &\rightarrow \widehat{\mathcal{Q}} = \mathcal{Q} + gA^\Lambda \mathcal{P}^0_\Lambda \\ \omega^{-x} &\rightarrow \widehat{\omega}^{-x} = \omega^{-x} + gA^\Lambda \mathcal{P}^{-x}_\Lambda \end{aligned} \tag{C.4}$$

where $\mathcal{P}^{-x}_\Lambda$ is given in eq. (A.8) and $\mathcal{P}^0_\Lambda$ is defined by the relation

$$i_\Lambda \mathcal{K} = -d\mathcal{P}^0_\Lambda \tag{C.5}$$

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[7]For the manifolds $ST(n)$ considered in the present paper we have $z^i = \{z^0, z^\alpha\} = S, Y^\alpha$, $\alpha = 1, \ldots n$

In computing the associated gauged curvatures we get:

$$\begin{aligned}
\widehat{\mathcal{K}} &= ig_{ij^*}\nabla z^i \wedge \nabla z^{j^*} + g\mathcal{F}^\Lambda \mathcal{P}^0_\Lambda \\
\widehat{\Omega}^{-x} &= \Omega^{-x}_{IJ}\nabla q^I \wedge \nabla q^J + g\mathcal{F}^\Lambda \mathcal{P}^{-x}_\Lambda
\end{aligned}$$
(C.6)

where

$$\nabla q^I = dq^I + gA^\Lambda k^I_\Lambda(q) \tag{C.7}$$

$k^I_\Lambda(q)$ being the quaternionic Killing vectors. We are now able to write down the full set of curvature definitions and rheonomic parametrizations of the N=2 matter coupled supergravity. We start with the hypermultiplets in the ungauged case. In the notation appearing in table 1 we have the positive and negative chirality hyperini $\zeta^{\overline{A}t}, \zeta^t_{\overline{A}}$. For the ungauged case we can write the following curvature definition for the right handed hyperino (a similar one holds for the other):

$$\nabla \zeta^{\overline{A}t} = d\zeta^{\overline{A}t} - \frac{1}{4}\gamma_{ab}\omega^{ab}\zeta^{\overline{A}t} - \Delta^{\overline{A}t}_{\overline{B}s}\zeta^{\overline{B}s} + \frac{i}{2}\mathcal{Q}\zeta^{\overline{A}t} \tag{C.8}$$

In the above equation $\Delta^{\overline{B}s\,\overline{A}t}$ is the $Sp(2m)$ connection. Indeed in our example the symplectic index α is splitted into an index $\overline{A}$ of $SU(2)_Q$ times an index t of $SO(m)$. The raising and lowering of the symplectic indices is realized by

$$C_{\alpha\beta} \equiv C_{\overline{A}t,\overline{B}s} = \epsilon_{\overline{BA}}\delta_{st}. \tag{C.9}$$

Moreover in eq. (C.8)

$$\gamma_{ab} = \frac{1}{2}[\gamma_a,\gamma_b] \equiv \begin{pmatrix} 2\sigma_{ab} & 0 \\ 0 & 2\overline{\sigma}_{ab} \end{pmatrix} \tag{C.10}$$

where we choose (in Minkowskian notation)

$$\begin{aligned}
\gamma^a &= \begin{pmatrix} 0 & \sigma^a \\ \overline{\sigma}^a & 0 \end{pmatrix} \\
(\overline{\sigma}^a)^{\dot\alpha\alpha} &= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^a)_{\beta\dot\beta}
\end{aligned}$$
(C.11)

with $\sigma^0 = \mathrm{diag}(-1,-1)$. The superspace parametrization of the quaternionic vielbein $u^A_{\overline{A}}{}^t$ is given by

$$u^A_{\overline{A}}{}^t = u^A_{a\overline{A}}{}^t V^a + \epsilon^{AB}\overline{\psi}_B\zeta^{\overline{B}t}\epsilon_{\overline{BA}} + \overline{\psi}^A\zeta^t_{\overline{A}} \tag{C.12}$$

Eq. (C.12) just fixes the supersymmetry transformation law of the quaternionic coordinate q^I. The rheonomic parametrization $\nabla\zeta^{\overline{A}t}$ compatible with the Bianchi identity coming from eq. (C.8) is the following one:

$$\nabla\zeta^{\overline{A}t} = \nabla_a\zeta^{\overline{A}t}V^a + iu^B_a{}_{\overline{B}}{}^s\gamma^a\psi^A\epsilon_{AB}\epsilon^{\overline{AB}}\delta_{st} \tag{C.13}$$

For the gauged case we have just to replace the ∇ derivative appearing in (C.8), which is covariant with respect to the spin, Kähler and $Sp(2m)$ connection with a derivative $\widehat{\nabla}$, covariant also with respect to the gauge connection. This substitution implies the following change in the rheonomic parametrization:

$$\widehat{\nabla}\zeta^{\overline{A}t} = \widehat{\nabla}\zeta^{\overline{A}t}_{old} + 2gu^A_{I\ \overline{A}}{}^t \epsilon^{\overline{BA}} k^I_\Lambda(q)\overline{L}^\Lambda\psi_A \tag{C.14}$$

The ungauged curvature definition of the gravitational sector are:

$$R^a = \mathcal{D}V^a - i\overline{\psi}_A \wedge \gamma^a\psi^A \tag{C.15}$$

$$\rho_A = d\psi_A - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi_A + \frac{i}{2}\mathcal{Q} \wedge \psi_A - \omega_A{}^B \wedge \psi_B \equiv \nabla\psi_A \tag{C.16}$$

$$\rho^A = d\psi^A - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi^A - \frac{i}{2}\mathcal{Q} \wedge \psi^A + \omega^B_A \wedge \psi^B \equiv \nabla\psi^A \tag{C.17}$$

$$R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^{ab} \tag{C.18}$$

where $\omega_A{}^B = 1/2i(\sigma_x)_A{}^B \omega^{-x}$ and $\omega^B_A = \epsilon^{AL}\omega_L{}^M \epsilon_{MB}$. For the vector multiplet we define, together with the differentials dz^i, $d\overline{z}^{i^*}$ ("curvatures" of z^i, $\overline{z}^{i^*}$), the following superspace field strengths:

$$\nabla\lambda^{iA} \equiv d\lambda^{iA} - \frac{1}{4}\gamma_{ab}\omega^{ab}\lambda^{iA} - \frac{i}{2}\mathcal{Q}\lambda^{iA} + \Gamma^i_j\lambda^{jA} + \omega^A_B \wedge \lambda^{iB} \tag{C.19}$$

$$\nabla\lambda^{i^*}_A \equiv d\lambda^{i^*}_A - \frac{1}{4}\gamma_{ab}\omega^{ab}\lambda^{i^*}_A + \frac{i}{2}\mathcal{Q}\lambda^{i^*}_A + \Gamma^{i^*}_{j^*}\lambda^{j^*}_A - \omega_A{}^B \wedge \lambda^{i^*}_B \tag{C.20}$$

$$F^\Lambda \equiv dA^\Lambda + \overline{L}^\Lambda\overline{\psi}_A \wedge \psi_B \epsilon^{AB} + L^\Lambda\overline{\psi}^A \wedge \psi^B \epsilon_{AB} \tag{C.21}$$

where Γ^i_j is the Levi–Civita connection and $L^\Lambda = e^{\mathcal{K}}X^\Lambda$.

The complete parametrizations of the curvatures, consistent with Bianchi identities following from eq.s (C.15)–(C.21), are given by

$$R^a = 0 \tag{C.22}$$

$$\rho_A = \rho_{A|ab}V^a \wedge V^b + \{(A_A^{|bB}\eta_{ab} + A'^{|b\ B}_A\gamma_{ab})\psi_B +$$
$$+ (\epsilon_{AB}T^+_{ab})\gamma^b\psi^B\} \wedge V^a - \frac{i}{4}\epsilon_{AB}\epsilon^{\overline{AB}}\overline{\zeta}^t_{\overline{A}}\gamma_{ab}\zeta^t_{\overline{B}}\gamma^b\psi^B V^a \tag{C.23}$$

$$\rho^A = \rho^A_{|ab}V^a \wedge V^b + \{(\overline{A}^{A|b}_B\eta_{ab} + \overline{A}'^{A|b}_B\gamma^{ab})\psi_B$$
$$+ (\epsilon^{AB}T^-_{ab})\gamma^b\psi_B\} \wedge V^a - \frac{i}{4}\epsilon^{AB}\epsilon_{\overline{AB}}\overline{\zeta}^{\overline{A}t}\gamma_{ab}\zeta^{\overline{B}t}\gamma^b\psi_B V^a \tag{C.24}$$

$$R^{ab} = R^{ab}_{\ \ cd}V^c \wedge V^d - i(\overline{\psi}_A\theta^{A|ab}_c + \overline{\psi}^A\theta^{ab}_{A|c}) \wedge V^c + \epsilon^{abcf}\overline{\psi}^A \wedge \gamma_f\psi_B(A'^B_{A|c} - \overline{A}'^B_{A|c})$$
$$+ i\epsilon^{AB}\overline{\psi}_A \wedge \psi_B T^{+ab} - i\epsilon_{AB}\overline{\psi}^A \wedge \psi^B T^{-ab} \tag{C.25}$$

$$F^\Lambda = F^\Lambda_{ab}V^a \wedge V^b + (if^\Lambda_i\overline{\lambda}^{iA}\gamma^a\psi^B\epsilon_{AB} + if^\Lambda_{i^*}\overline{\lambda}^{i^*}_A\gamma^a\psi_B\epsilon^{AB}) \wedge V_a \tag{C.26}$$

$$\nabla\lambda^{iA} = \nabla_a\lambda^{iA}V^a + iZ^i_a\gamma^a\psi^A + G^{+i}_{ab}\gamma^{ab}\psi_B\epsilon^{AB} + Y^{iAB}\psi_B \tag{C.27}$$

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$$\nabla \lambda_A^{i^*} = \nabla_a \lambda_A^{i^*} V^a + i\overline{Z}_a^{i^*} \gamma^a \psi_A + G_{ab}^{-i^*} \gamma^{ab} \psi^B \epsilon_{AB} + Y^{i^*}{}_{AB} \psi^B \tag{C.28}$$

$$dz^i = Z_a^i V^a + \overline{\lambda}^{iA} \psi_A \tag{C.29}$$

$$d\overline{z}^{i^*} = Z_a^{i^*} V^a + \overline{\lambda}_A^{i^*} \psi^A \tag{C.30}$$

where

$$A_{A\,|a}{}^B = -\frac{i}{4} g_{k^*\ell}(\overline{\lambda}_A^{k^*} \gamma_a \lambda^{\ell B} - \delta_A^B \overline{\lambda}_C^{k^*} \gamma_a \lambda^{\ell C}) \tag{C.31}$$

$$A'^{B}_{A\,|a} = \frac{i}{4} g_{k^*\ell}(\overline{\lambda}_A^{k^*} \gamma_a \lambda^{\ell B} - \frac{1}{2}\delta_A^B \overline{\lambda}_C^{k^*} \gamma_a \lambda^{C\ell}) + \frac{i}{4}\delta_A^B \overline{\zeta}_{\overline{A}}^t \gamma_a \zeta^{\overline{A}t} \tag{C.32}$$

$$S_{AB} = \overline{S}^{AB} = 0 \tag{C.33}$$

$$\theta_{cA}^{ab} = 2\gamma^{[a} \rho_A^{b]c} + \gamma^c \rho_A^{ab}; \quad \theta_c^{ab\,A} = 2\gamma^{[a} \rho^{b]c|A} + \gamma^c \rho^{ab|A} \tag{C.34}$$

$$T_{ab}^+ = 2\mathrm{iIm}\mathcal{N}_{\Lambda\Sigma}L^\Sigma(\mathcal{F}_{ab}^{\Lambda+} + \frac{1}{8}\nabla_i f_j^\Lambda \overline{\lambda}^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB} - \frac{1}{4}\epsilon_{\overline{AB}} \overline{\zeta}^{\overline{A}t} \gamma_{ab} \zeta^{\overline{B}t} L^\Lambda)$$

$$T_{ab}^- = 2\mathrm{iIm}\mathcal{N}_{\Lambda\Sigma}\overline{L}^\Sigma(\mathcal{F}_{ab}^{\Lambda-} + \frac{1}{8}\nabla_{i^*} \overline{f}_{j^*}^\Lambda \overline{\lambda}_A^{i^*} \gamma_{ab} \lambda_B^{j^*} \epsilon^{AB} - \frac{1}{4}\epsilon^{\overline{AB}} \overline{\zeta}_{\overline{A}}^t \gamma_{ab} \zeta_{\overline{B}}^t \overline{L}^\Lambda) \tag{C.35}$$

$$G_{ab}^{i+} = -g^{i^*j}\overline{f}_{j^*}^\Gamma \mathrm{Im}\mathcal{N}_{\Gamma\Lambda} \left(\mathcal{F}_{ab}^{\Lambda+} + \frac{1}{8}\nabla_i f_j^\Lambda \overline{\lambda}^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB} - \frac{1}{4}\epsilon_{\overline{AB}} \overline{\zeta}^{\overline{A}t} \gamma_{ab} \zeta^{\overline{B}t} L^\Lambda\right)$$

$$G_{ab}^{i^*-} = -g^{i^*j}f_j^\Gamma \mathrm{Im}\mathcal{N}_{\Gamma\Lambda} \left(\mathcal{F}_{ab}^{\Lambda-} + \frac{1}{8}\nabla_{i^*} \overline{f}_{j^*}^\Lambda \overline{\lambda}_A^{i^*} \gamma_{ab} \lambda_B^{j^*} \epsilon^{AB} - \frac{1}{4}\epsilon^{\overline{AB}} \overline{\zeta}_{\overline{A}}^t \gamma_{ab} \zeta_{\overline{B}}^t \overline{L}^\Lambda\right)$$

$$\tag{C.36}$$

$$Y^{ABi} = g^{ij^*}C_{j^*k^*\ell^*} \overline{\lambda}_C^{k^*} \lambda_D^{\ell^*} \epsilon^{AC} \epsilon^{BD} \tag{C.37}$$

$$Y_{AB}^{i^*} = g^{i^*j}C_{ik\ell} \overline{\lambda}^{kC} \lambda^{\ell D} \epsilon_{AC} \epsilon_{BD} \tag{C.38}$$

The special geometry gadgets $L^\Lambda, \overline{L}^\Lambda, f_i^\Lambda, f_{i^*}^\Lambda$ and the tensors C_{ijk}, and $C_{i^*j^*k^*}$ turn out to be constrained by consistency of the Bianchi identities as it follows

$$\nabla_{i^*}L^\Lambda = \nabla_i \overline{L}^\Lambda = 0 \tag{C.39}$$

$$f_i^\Lambda = \nabla_i L^\Lambda; \quad f_{i^*}^\Lambda = \nabla_{i^*} L^\Lambda \tag{C.40}$$

$$\nabla_{\ell^*}C_{ijk} = \nabla_\ell C_{i^*j^*k^*} = 0 \tag{C.41}$$

$$\nabla_{[\ell}C_{i]jk} = \nabla_{[\ell^*}C_{i^*]j^*k^*} = 0 \tag{C.42}$$

$$ig^{i\ell^*}f_{\ell^*}^\Lambda C_{ijk} = \nabla_j f_k^\Lambda \tag{C.43}$$

We do not report the explicit calculation to prove the above equations, but we stress that they are fully determined by the Bianchi identities of N=2 supergravity. The solution for C_{ijk} can be expressed by ([36])

$$C_{ijk} = 2\mathrm{iIm}\mathcal{N}_{\Lambda\Sigma} f_i^\Lambda \nabla_j f_k^\Sigma \tag{C.44}$$

In the gauged case we have firstly to replace in the curvature definitions ∇ with $\widehat{\nabla}$, namely the derivative covariant with respect to the gauge field. Secondly, the new parametrization will contain extra terms with respect to the old ones which are proportional to the gauge coupling constant g. In particular the new parametrization are:

$$R^a = 0 \tag{C.45}$$

$$\widehat{\rho}_A = \widehat{\rho}_A^{(old)} + igS_{AB}\gamma_a\psi^B \wedge V^a \tag{C.46}$$

$$\widehat{\rho}^A = \widehat{\rho}^{A\,(old)} + ig\overline{S}^{AB}\gamma_a\psi_B \wedge V^a \tag{C.47}$$

$$\widehat{R}^{ab} = \widehat{R}^{ab\,(old)} - \overline{\psi}_A \wedge \gamma^{ab}\psi_B g\overline{S}^{AB} - \overline{\psi}^A \wedge \gamma^{ab}\psi^B gS_{AB} \tag{C.48}$$

$$F^\Lambda = F^{\Lambda\,(old)} \tag{C.49}$$

$$\widehat{\nabla}\lambda^{iA} = \widehat{\lambda}^{iA\,(old)} + gW^{iAB}\psi_B \tag{C.50}$$

$$\widehat{\lambda}_A^{i^*} = \widehat{\lambda}_A^{i^*\,(old)} + gW_{AB}^{i^*}\psi^B \tag{C.51}$$

$$\widehat{\nabla}z^i = \nabla z^{i\,old} \quad \widehat{\nabla}z^{i^*} = \nabla z^{i^*\,(old)} \tag{C.52}$$

together with equation (C.8) for the hyperinos. In eq. (C.52) S_{AB} and the corresponding conjugated expression is given by:

$$S_{AB} = \frac{1}{2}i(\sigma_x)_A^{\ C}\epsilon_{BC}\mathcal{P}_\Lambda^x L^\Lambda$$

$$\overline{S}^{AB} = \frac{1}{2}i(\sigma_x)_C^{\ B}\epsilon^{CA}\mathcal{P}_\Lambda^x \overline{L}^\Lambda \tag{C.53}$$

while W^{iAB} is given by the sum of a symmetric part plus an antisymmetric one, where

$$W^{i[AB]} = \epsilon^{AB}k_\Lambda^i \overline{L}^\lambda$$

$$W^{i(AB)} = -i(\sigma_x)_C^{\ B}\epsilon^{CA}\mathcal{P}_\Sigma^x g^{ij^*}f_{j^*}^\Sigma \tag{C.54}$$

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